Well known and Little known Nation

One person's perspective

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Algebras and Lattices in Hawaii, May 24 2018

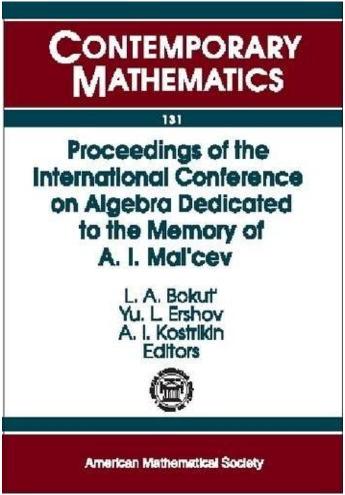
Outline

1. From Novosibirsk to Nashville
   - Novosibirsk 1989
   - 90s
   - Nashville, 2002
2. Hawai'i and quasivarieties
   - 2004
   - Representations
   - Without equality
3. 2010s
   - Closure operators and implicational bases
   - AAB workshops
   - Little known

Outline

Malcev Conference 1989

Academ town near Novosibirsk



Proceedings: Preface

In August 1989, more than 700 Soviet algebraists and more than 200 foreign mathematicians convened in Novosibirsk in the former Soviet Union for the International Conference on Algebra. Dedicated to the memory of A. I. Mal'cev, the great Russian algebraist and logician, the conference marked the first time since the International Congress of Mathematicians was held in Moscow in 1966 that Soviet algebraists could meet with a large number of their foreign colleagues. ... The papers span a broad range of areas including groups, Lie algebras, associative and nonassociative rings, fields and skew fields, differential algebra, universal algebra, categories, combinatorics, logic, algebraic geometry, topology, and mathematical physics.

Birkhoff-Malcev Problem and lattices of subsemilattices

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Describe lattices that can be represented as lattice of sub-quasivarieties (sub-varieties) of some qiasivariety (variety) of algebraic systems.

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For any quasivariety of algebraic systems $\mathcal{K}$, there exists an algebraic lattice $\mathbf{A}$ *and a quasi-order σ on A such that the lattice $\mathcal{L}_q(\mathcal{K})$ of subquasivarieties of $\mathcal{K}$ is represented as* $\mathbf{S_p}(\mathbf{A}, \sigma)$, *the lattice of algebraic subsets of* $\mathbf{A}$ *closed under σ. Every lattice* $\mathbf{S_p}(\mathbf{A})$ *is isomorphic to $\mathcal{K}$ of a quasivariety of predicate systems.*

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In cases when $\mathbf{A}$ can be chosen finite, the representation becomes $\mathbf{Sub}(\mathbf{A}, \wedge, \mathbf{1}, \sigma)$.

Quick time travel forward

- Almost 30 years forward, we replace $\mathbf{S_p}(\mathbf{A}, \sigma)$ by $\mathbf{S_p}(\mathbf{A}, H)$, where H is set of operators on $\mathbf{A}$ preserving arbitrary meets and joins of non-empty chains, and $\mathbf{S_p}(\mathbf{A}, H)$ is the structure of algebraic subsets of $\mathbf{A}$ closed under H.
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- About mid-point of this development we thought of $\mathbf{S_p}(\mathbf{A},H)$ in dual form, as $\mathbf{Con}(\mathbf{S},\vee,\mathbf{0},G)$, the lattices of congruences of a semilattice $\mathbf{S}$ with monoid of operators G. It was a switch in thinking about quasivarieties as their quasi-equational theories.

Quick time travel into past: Zipper condition

Theorem (Bill Lampe, AU 86)

Every lattice $\mathbf{L}$ *representable as* $\mathcal{L}(\Sigma)$*, the lattice of equational theories extending a given equational theory* Σ*, satisfies the following condition:*

$$\text{(Zipper)} \qquad \text{for every } a, c \in L, \text{ and every } B \subseteq L, \text{ if } \bigvee B = 1 \text{ and}$$

$$a \wedge b = c \text{ for all } b \in B, \text{ then } a = c.$$

Novosibirsk in 1989

Theorem (K.Adaricheva, 1991)

A finite atomistic lattice $\mathbf{L}$ *is represented as* $\mathbf{Sub}(\mathbf{A}, \wedge, \mathbf{1},)$ *iff it satisfies the following properties:*

(1) *the sum of two atoms contains no more than 3 atoms;*

(2) *there is no sequence of atoms* $a_0, a_1, \dots a_n = a_0$*, where* a_{i+1}*, with index computed modulo* n*, is contained in the join of* a_i *and another atom* b_i*;*

(3) *and (4) and (5): more technical properties.*

Hawai'i before 1991

- Ralph, JB and Jaroslav Ježek worked on monograph "Free lattices";
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Hawai'i before 1991

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- finite lower bounded lattices, which are homomorphic images of lower bounded homomorphisms from a free lattice are described as lattices without D-cycles;
- relying on earlier essential ideas of lower bounded homomorphism and D-relation developed in papers by R. McKenzie (1972) and B. Jónsson and J.B.Nation (1977).

Malcev conference

Theorem (K.Adaricheva and V. Gorbunov, 1989)

Every lattice $\mathbf{L}$ *representable as* $\mathcal{L}_q(\mathcal{K})$ *admits an equaclosure operator* $\nu : \mathbf{L} \to \mathbf{L}$*, which allows to describe all lattices of quasivarieties that may occur within the class* $\mathbf{Co}(\mathbf{P})$*,* $\mathbf{P}$ *a poset.*

Outline

In the 90s

- V. Gorbunov gives a talk at B. Jonsson's conference, 1990;
- "Free lattices" published, 1991;
- Alan Day conference, 1992;

In the 90s: continued

R. Freese, K. Kearnes and JB Nation publish paper to 80th birthday of G. Birkhoff, 1995.

Theorem

If the type of quasivariety $\mathcal{K}$ has only finitely many relational symbols, then the lattice $\mathcal{L}_q(\mathcal{K})$ satisfies the following quasi-identity:

$$\&_{0<i<n}(x_i \leq x_{i+1} \vee y_i \;\&\; x_i \wedge y_i \leq x_{i+1}) \;\&\; x_0 \wedge \cdots \wedge x_{n-1} = 0 \to x_0 = 0$$

The presence of an equa-closure operator plays essential role in the proof.

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- “Hotel Slavia Blues” was included into the book “Mathematical songs of Brian Davey”, with the fine print that it is written by JB, 2015.

Viktor Gorbunov: 1999

Memorial issue of Algebra Universalis: 2001, KA and W. Dziobiak, guest editors.

Theorem (JB Nation, AU 2001)

Let L be a finite lattice that satisfies $SD_\wedge$. Then L fails $SD_\vee$ iff there exists B-cycle $aBbBa$, for some $a, b \in J(L)$.

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- most likely reviewer of the paper was ...

Largest extension of a finite convex geometry

Definition

A closure system (X, ϕ) satisfies the *anti-exchange property* if for all $x \neq y$ and all closed sets $Y \subseteq X$,

$$x \in \phi(Y \cup \{z\}) \text{ and } x \neq z, x \notin Y \text{ imply that } z \notin \phi(Y \cup \{x\}). \quad (1)$$

Definition

A closure system that satisfies the anti-exchange property is called a *convex geometry*.

Example

If A is an algebraic lattice and $S_p : 2^A \to 2^A$ is an operator generating a smallest algebraic subset for any input $Y \subseteq A$, then (A, S_p) is a convex geometry.

Largest extension of a finite convex geometry

Definition

A closure space (X, Δ) is a *(strong) extension* of (X, ϕ), if $Cl(X, \phi)$ is a sublattice of $Cl(X, \Delta)$.

Theorem (KA and JB Nation, AU 2005)

Every finite join semidsitributive lattice has a largest join semidistributive extension. This largest extension is atomistic, and hence the closure lattice of a convex geometry.

Fast time forward: H. Yoshikawa, H. Hirai and K. Makino, in “A representation of antimatroids by Horn rules in its application to educational systems”, Journal of Math. Psychology, 2017:

Problem

Find effective algorithmic solution to obtain largest antimatroid extension given a set of rules for an antimatroid.

Outline

Hawai'i 2004

Lemma (R. Freese and J.B.Nation, Pacific J. Math. 73)

Let $\mathbf{A}$ *be a finite lower semilattice with greatest element* 1_A*, and let* $\mathbf{Sub}_\wedge\mathbf{A}$ *be the lattice of its subsemilattices containing* 1_A*. If* $\mathbf{S}_\vee$ *is the upper semilattice induced on* A *by the order relation of* $\mathbf{A}$*, then* $\mathbf{Con}S_\vee$ *is dually isomorphic to* $\mathbf{Sub}_\wedge A$*.*

Hawai'i 2004 continued

Theorem (Fajtlowicz and J. Schmidt, 76)

Let $\mathbf{A}$ *be an algebraic lattice and* $\mathbf{S}$ *denotes the join semilattice (with 0) of its compact elements. Then* $\mathbf{S_p}(\mathbf{A}) \cong^d \mathbf{Con\ S}$.

Congruence lattices of semilattices with operators

Fast time forward:

Theorem (J. Hyndman, J.B.Nation, J. Nishida, Studia Logica 16)

Let $\mathbf{A}$ be an algebraic lattice with the monoid H of algebraic operators, and $\mathbf{S}$ denotes the join semilattice (with 0) of its compact elements. Then there exists a monoid of endomorphisms of $\mathbf{S}$ such that $\mathbf{S_p}(\mathbf{A}, H) \cong^d \mathbf{Con}\ (\mathbf{S}, \vee, \mathbf{0}, G)$.

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Corollary (1) (KA and JB Nation, IJAC 12)

Equa-closure operator ν *(or, in dual form, equa-interior operator) on a quasivariety lattice should satisfy property:*

$$(\dagger) \qquad \nu(x \wedge \tau(x \vee z)) \geq x \wedge \tau x$$

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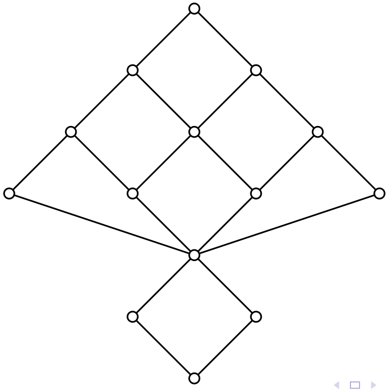
Representations

Corollary (2) (KA and JB Nation, IJAC 12)

The near-leaf lattice on the picture is isomorphic to a quasivariety lattice with equality, therefore finite lattice of quasivarieties are not necessarily lower bounded.



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Quasivarieties without equality

Theorem (JB Nation, Notre Dame J. Formal Logic, 13)

If $(\mathbf{S}, \mathbf{0}, \vee, G)$ *is a semilattice with operators, then* **Con** $(\mathbf{S}, \vee, \mathbf{0}, G)$ *is isomorphic to a lattice of quasi-equational theories in the language that may not contain equality.*

Outline

Pre-cursors

Fast time travel backward.

Theorem (V. Gorbunov and V. Tumanov, Algebra and Logic, 80)

For any algebraic lattice $\mathbf{A}$, *the lattice* $\mathbf{S_p}(\mathbf{A})$ *is isomorphic to lattices of quasivarieties of one-element structures in language with unary predicates.*

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Definition (JB Nation, AU 1990)

OD-graph of finite lattice $\mathbf{L}$ is a structure $(J, \leq, \mathcal{C})$, where $J = Ji(L)$, $\mathcal{C} : J \to 2^{2^J}$ and $\mathcal{C}(j) = \{C : C$ is a minimal cover of $j\}$.

Lemma

Every finite lattice $\mathbf{L}$ *is isomorphic to the lattice of down-sets* $X \subseteq Ji(\mathbf{L})$ *closed with respect to the following rule: if* $C \in \mathcal{C}(j)$ *and* $C \subseteq X$ *then* $j \in X$.

Prompt

K. Bertet, B. Monjardet *The multiple facets of canonical direct implicational basis*, TCS, 2010.

Theorem (KA, JB Nation, R. Rand, DAM 2013)

- *OD-graph is an impicational basis (the D-basis) of a closure system associated with a finite lattice.*
- *The D-basis is a subset of a canonical direct basis.*
- *The D-basis is ordered direct (allows fast computation of closures).*

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Theorem (KA and JB Nation, DAM 2014)

There exists a polynomial algorithm that, given canonical basis of Guigue-Duquenne, identifies whether the closure system defined by the basis does not have D-cycles.

Outline

AAB workshops

Starting in 2010, 4 workshops were organized on the topic of overlaps between lattice theory, Horn logic, Horn Boolean functions, closure operators and directed hypergraphs. The largest one ran in Dagstuhl in 2014.

Applications

Theorem (KA and JB Nation, TCS 2017)

The D-basis of Galois lattice associated with a binary table can be computed by polynomial reduction to hypergraph dualization problem.

Lemma (Freese, Jezek, Nation, *Free Lattices*, Chapter 11)

For distinct join irreducible elements a, b of a finite lattice aDb iff $a \nearrow q \searrow b$ for some meet irreducible element q.

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Little known Nation

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- He is a creator of LUST algorithm to identify common metagenes across 16 types of cancer.
- Dbasis algorithm is implemented in code for retrieval of implicational basis from large tables (first testing was done in Kazakhstan).

Little known continued

"Measuring the implications of the D-basis in analysis of data in biomedical studies" (Proceedings of ICFCA, 2015) was done in collaboration of Cancer center of UofH and Biology department of University in Astana, Kazakhstan;

More of little known

The new paper on retrieval of rules of high confidence is just accepted at Data Mining conference DTMN in Sydney (July 2018). Co-authors: Oren Segal (Hofstra, CS), Justin Cabot-Miller (undegrad math/CS major), Anuar Sharafudinov (AILabs in Kazakhstan).

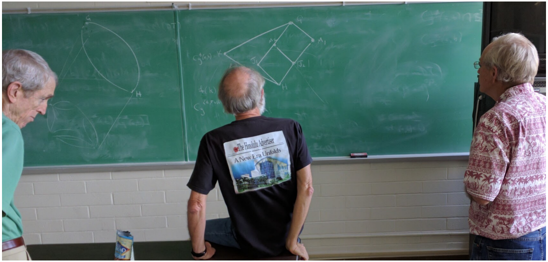
JB and Jen Hyndman are about to publish their monograph:
The lattice of subquasivarieties of a locally finite quasivariety

More known

JB had been working and continues to work on

- projective planes (check one of next talks today);
- whales;
- reflection group codes and their decoding;
- inherently non-finitely based varieties;
- infinite convex geometries;
- long distance running;
- volunteering for a soccer league;
- playing a trumpet on important occasions...

Hawaiian nation of universal algebraists



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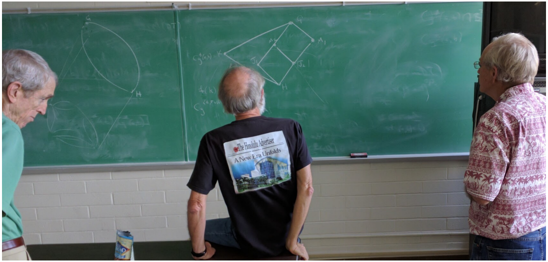

MAHALO!