Splitting residuated (semi)lattices

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Dedicated to Alden, Ralph M., Ralph F., Bill and JB, who one way ore another shaped me as a mathematician.

Splitting algebras

1. if $\mathcal{K}$ is a variety and $\mathcal{L}_{\mathcal{K}}$ is its lattices of subvarieties, a **splitting pair** in $\mathcal{L}_{\mathcal{V}}$ is a pair of subvarieties $(\mathcal{W}, \mathcal{V})$ such that for every $\mathcal{K}' \subseteq \mathcal{K}$ either $\mathcal{V} \subseteq \mathcal{K}'$ or $\mathcal{K}' \subseteq \mathcal{W}$;
2. we can prove: $\mathcal{W}$ is one-based and $\mathcal{V}$ is generated by a single finitely generated subdirectly irreducible algebra $\mathbf{A}$;
3. if $\mathcal{K}$ has the finite model property, then $\mathbf{A}$ must be finite;
4. if $\mathcal{K}$ is also congruence distributive, then $\mathbf{A}$ is uniquely determined (by Jònsson's Lemma);
5. if for $\mathbf{A} \in \mathcal{K}$ there exists a $\mathcal{W}_{\mathbf{A}} \subseteq \mathcal{K}$ such that $(\mathcal{W}_{\mathbf{A}}, \boldsymbol{V}(\mathbf{A}))$ is a splitting pair, then $\mathbf{A}$ is **splitting algebra** in $\mathcal{K}$; clearly $\mathbf{A}$ can be assumed to be s.i. and finitely generated.
6. if $\mathbf{A}$ is splitting then any equation axiomatizing $\mathcal{W}_{\mathbf{A}}$ is the **splitting equation** of $\mathbf{A}$ and $\mathcal{W}_{\mathbf{A}}$ is the **conjugate variety** of $\mathbf{A}$.

- The theorem hinted in the previous slide does not say that if the hypotheses hold, then there is a splitting algebra in $\mathcal{V}$; nor its proof produces an effective way of determining the splitting equation of $\mathbf{A}$, in case it is splitting. Both the existence and the splitting equation require *ad hoc* arguments.

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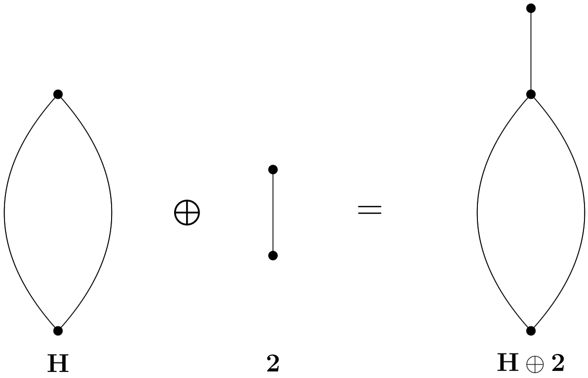
- R. McKenzie [1972] did exactly that, characterizing the splitting algebras in the variety of all lattices (that are congruence distributive and have the finite model property) and also giving an algorithm to find their splitting equations.
- Almost at the same time in a different part of the world V. Jankov was studying intermediate logics, i.e. subvarieties of the variety $\mathcal{HA}$ of Heyting algebras.

Jankov's result

Jankov found a way to associate to any finite subdirectly irreducible Heyting algebra $\mathbf{A}$ a term $J_{\mathbf{A}}$ (called the *Jankov formula*) and was able to prove essentially that:

- the largest variety of Heyting algebras not containing $\boldsymbol{V}(\mathbf{A})$ is axiomatized by $J_{\mathbf{A}} \approx 1$;
- hence any finite subdirectly irreducible Heyting algebra is splitting in $\mathcal{HA}$ with splitting equation $J_{\mathbf{A}} \approx 1$.

Subdirectly irreducible Heyting algebras



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- This is exactly the statement we would like to generalize.

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- a proper context;
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- some knowledge of the subdirectly irreducible algebras in the context.

The context: FL-algebras and residuated semilattices

- An algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \to, \leftarrow, 0, 1\rangle$ is an **FL-algebra** if
  1. $\langle A, \vee, \wedge\rangle$ is a lattice;
  2. $\langle A, \cdot, 1\rangle$ is a monoid;
  3. $\to$ and $\leftarrow$ are the left and right residuation w.r.t. $\cdot$;
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- By $\mathcal{FL}_{ew}$ we denote the variety of commutative, integral and zero-bounded FL-algebras.
- The variety of FL-algebras and the variety of residuated semilattices are congruence distributive (the first is obvious, the second less obvious but still true).

Ordinal sums: semilattices

Let $\mathbf{F}, \mathbf{S}$ be two integral residuated semilattices. The **ordinal sum** $\mathbf{F} \oplus \mathbf{S}$ is $F \cup S$ with the operations defined in the following way. If x, y both belong to F or S then the operations are defined as those in each algebra; otherwise

$$x \to y = x \leftarrow y = \begin{cases} y & \text{if } x \in S \text{ and } y \in F \\ 1 & \text{if } x \in F \setminus \{1\} \text{ and } y \in S \end{cases}$$

$$x \cdot y = \begin{cases} y & \text{if } x \in S \text{ and } y \in F\{1\} \\ x & \text{if } x \in F \setminus \{1\} \text{ and } y \in S \end{cases}$$

$$x \wedge y = \begin{cases} y & \text{if } x \in F \text{ and } y \in S \setminus \{1\} \\ x & \text{if } x \in S \setminus \{1\} \text{ and } y \in F \end{cases}$$

It is easily seen that $\mathbf{F} \oplus \mathbf{S}$ is always a residuated semilattice and the ordering is the one obtained stacking the two semilattices one over the other.

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4. In the second case if m is the minimum of $\mathbf{B}$

$$x \vee y = \begin{cases} x \vee^{\mathbf{B}} y & \text{if } x, y \in B \\ x \vee^{\mathbf{A}} y & \text{if } x, y \in A \text{ and } x \vee y < 1 \\ m & \text{if } x, y \in A \text{ and } x \vee y = 1 \\ x & \text{if } x \in B \text{ and } y \in A \\ y & \text{if } x \in A \text{ and } y \in B; \end{cases}$$

A look at Jankov's argument

1. If $\mathbf{S}$ is an integral residuated semilattice and $p_i(\mathbf{x})$, $i = i, \dots, n$ are residuated semilattices terms , then the equation $\bigwedge_{i=1}^n p_i(\mathbf{x}) \approx 1$ is true in a model if and only if all the equations $p_i(\mathbf{x}) \approx 1$ are true in the model.

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5. If $S = \{a_1, \dots, a_n, \star, 1\}$ the **t-diagram**) for **S** is

   $$T(X_S) = \bigwedge\{x_{u*v} \leftrightarrow x_u * x_v : u, v \in S, * \in \{\vee, \wedge, \rightarrow, \leftarrow, \cdot, 1\}\}.$$

   Observe that $T(X_S)$ encodes all the operation tables of **S** and $T(a_1, \dots, a_n, \star, 1) = 1$ by design.

Theorem

Let $\mathbf{S}$ *be a finite subdirectly irreducible integral residuated semilattice* $S = \{a_1, \dots, a_n, \star, 1\}$ *and let* $\mathbf{B}$ *any integral residuated semilattice. Then* $\mathbf{S} \in \boldsymbol{IS}(\mathbf{B})$ *if and only if there are* $b_1, \dots, b_n, b_\star \in B$ *with* $b_\star \neq 1$ *and*

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3. Next if $\mathbf{S} \in \boldsymbol{V}(\mathbf{B})$, then $\mathbf{B} \not\models J_T(X_S) \approx 1$, since **S** does not.
4. So if **S** is a finite subdirectly irreducible algebra, $\mathbf{S} \in \mathcal{U}$ and $\mathcal{W}$ is the subvariety of $\mathcal{U}$ axiomatized by $J_T(X_S) \approx 1$, then $\mathbf{S} \notin \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{W}^{\mathcal{U}}_{\mathbf{S}}$.

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7. For instance T. Kowalski and H. Ono [2000] showed that if $\mathcal{U} = \mathcal{FL}_{ew}$, then $\mathcal{W} = \mathcal{W}^{\mathcal{U}}_{\mathbf{S}}$ if and only if $\mathbf{S} = \mathbf{2}$.

GBL-algebras

- An FL-algebra or a residuated semilattice **S** is **divisible** if the underlying ordering is the inverse divisibility ordering, i.e. all $a, b \in S$

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- Both $\mathcal{H}$ and $\mathcal{GBL}_{ew}$ have the finite model property.

Subdirecly irreducible GBL-algebras and hoops

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- Any subdirectly irreducible algebra in $\mathcal{GBL}_{ew}$ is of the form $\mathbf{F} \oplus \mathbf{S}$, where $\mathbf{S}$ is a totally ordered Wajsberg hoop [Jipsen-Montagna, 2008].

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- Define

$$\widehat{J}(X_A) = D(X_A) \to (T(X_A) \to x_\star);$$

  the Jankov formula we are going to use is $\hat{J}(X_A) \approx 1$.

- The rationale: the original request that $T(X_A) \to x_\star \approx 1$ was too strong, so we have to settle for less. Hence we ask that $T(X_A) \to x_\star$ be large enough to be above $D(X_A)$.

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- Roughly speaking we consider the algebras in $\mathcal{GBL}_{ew}$ that have the following property: if they are generated by the same number of elements as $\mathbf{A}$ and they have a unique maximal idempotent coatom, then not all the operation tables are encoded by the t-diagram of $\mathbf{A}$. Note that $\mathbf{A} \not\models \widehat{J}(X_A) \approx 1$ by design.

The results

Theorem

A subdirectly irreducible algebra in $\mathcal{GBL}_{ew}$ is splitting if and only if it is $\mathbf{F} \oplus \mathbf{2}$ for some finite $\mathbf{F} \in \mathcal{GBL}_{ew}$.

Theorem

A subdirectly irreducible hoop $\mathbf{A}$ is splitting in the variety of hoops if and only if $A = \mathbf{F} \oplus \mathbf{2}$ for some finite hoop $\mathbf{F}$.

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- They are both known to have the finite model property [?] and their splitting algebras can be completely characterized.
- We can also try to dispose of divisibility altogether, and this is the subject of our ongoing research.