Splitting residuated (semi)lattices

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Dedicated to Alden, Ralph M., Ralph F., Bill and JB, who one way or another shaped me as a mathematician.
if $\mathcal{K}$ is a variety and $\mathcal{L}_\mathcal{K}$ is its lattices of subvarieties, a **splitting pair** in $\mathcal{L}_\mathcal{V}$ is a pair of subvarieties $(\mathcal{W}, \mathcal{V})$ such that for every $\mathcal{K}' \subseteq \mathcal{K}$ either $\mathcal{V} \subseteq \mathcal{K}'$ or $\mathcal{K}' \subseteq \mathcal{W}$;

we can prove: $\mathcal{W}$ is one-based and $\mathcal{V}$ is generated by a single finitely generated subdirectly irreducible algebra $A$;

if $\mathcal{K}$ has the finite model property, then $A$ must be finite;

if $\mathcal{K}$ is also congruence distributive, then $A$ is uniquely determined (by Jònsson’s Lemma);

if for $A \in \mathcal{K}$ there exists a $\mathcal{W}_A \subseteq \mathcal{K}$ such that $(\mathcal{W}_A, V(A))$ is a splitting pair, then $A$ is splitting algebra in $\mathcal{K}$; clearly $A$ can be assumed to be s.i. and finitely generated.

if $A$ is splitting then any equation axiomatizing $\mathcal{W}_A$ is the splitting equation of $A$ and $\mathcal{W}_A$ is the conjugate variety of $A$. 
The theorem hinted in the previous slide does not say that if the hypotheses hold, then there is a splitting algebra in $\mathcal{V}$; nor its proof produces an effective way of determining the splitting equation of $\mathbf{A}$, in case it is splitting. Both the existence and the splitting equation require \textit{ad hoc} arguments.
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R. McKenzie [1972] did exactly that, characterizing the splitting algebras in the variety of all lattices (that are congruence distributive and have the finite model property) and also giving an algorithm to find their splitting equations.
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Almost at the same time in a different part of the world V. Jankov was studying intermediate logics, i.e. subvarieties of the variety $\mathcal{HA}$ of Heyting algebras.
Jankov’s result

Jankov found a way to associate to any finite subdirectly irreducible Heyting algebra $A$ a term $J_A$ (called the Jankov formula) and was able to prove essentially that:

- the largest variety of Heyting algebras not containing $V(A)$ is axiomatized by $J_A \approx 1$;
- hence any finite subdirectly irreducible Heyting algebra is splitting in $HA$ with splitting equation $J_A \approx 1$. 
Subdirectly irreducible Heyting algebras

The ordinal sum.

H \oplus 2 = H \oplus 2

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**Theorem**

*A finite Heyting algebra is splitting if and only if it is of the form* $H \oplus 2$ *for some Heyting algebra* $H$.  

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- This is exactly the statement we would like to generalize.
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- some knowledge of the subdirectly irreducible algebras in the context.
An algebra $A = \langle A, \lor, \land, \cdot, \rightarrow, \leftarrow, 0, 1 \rangle$ is an FL-algebra if

1. $\langle A, \lor, \land \rangle$ is a lattice;
2. $\langle A, \cdot, 1 \rangle$ is a monoid;
3. $\rightarrow$ and $\leftarrow$ are the left and right residuation w.r.t. $\cdot$;
4. $0$ is an element of $A$. 

A residuated lattice is a sub-reduct of an FL-algebra to the type without $0$. A residuated semilattice is a sub-reduct of an FL-algebra to the type without $0$ and $\lor$. An FL-algebra or a residuated (semi)lattice is commutative if satisfies $xy \approx yx$, integral if satisfies $x \leq 1$ and zero-bounded if satisfies $0 \leq x$; they all form varieties. By FL we denote the variety of commutative, integral and zero-bounded FL-algebras. The variety of FL-algebras and the variety of residuated semilattices are congruence distributive (the first is obvious, the second less obvious but still true).
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Splitting residuated (semi)lattices
The context: FL-algebras and residuated semilattices

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- An FL-algebra or a residuated (semi)lattice $\mathbf{A}$ is **commutative** if it satisfies $xy \approx yx$, **integral** if it satisfies $x \leq 1$ and **zero-bounded** if it satisfies $0 \leq x$; they all form varieties.

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- The variety of FL-algebras and the variety of residuated semilattices are congruence distributive (the first is obvious, the second less obvious but still true).
Ordinal sums: semilattices

Let $F, S$ be two integral residuated semilattices. The ordinal sum $F \oplus S$ is $F \cup S$ with the operations defined in the following way. If $x, y$ both belong to $F$ or $S$ then the operations are defined as those in each algebra; otherwise

$x \rightarrow y = x \leftarrow y = \begin{cases} y & \text{if } x \in S \text{ and } y \in F \\ 1 & \text{if } x \in F \setminus \{1\} \text{ and } y \in S \end{cases}$

$x \cdot y = \begin{cases} y & \text{if } x \in S \text{ and } y \in F \{1\} \\ x & \text{if } x \in F \setminus \{1\} \text{ and } y \in S \end{cases}$

$x \land y = \begin{cases} y & \text{if } x \in F \text{ and } y \in S \setminus \{1\} \\ x & \text{if } x \in S \setminus \{1\} \text{ and } y \in F \end{cases}$

It is easily seen that $F \oplus S$ is always a residuated semilattice and the ordering is the one obtained stacking the two semilattices one over the other.
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The ordering that results in stacking two integral residuated lattices one over another might not be a lattice ordering.

If 1 is not join irreducible in $A$ and $B$ has no minimum, then for $a, b \in A$ with $a \vee_A b = 1$, then $a \vee b$ simply does not exist, since the upper bounds of $a, b$ are all the elements of $B$ and $B$ has no minimum.

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In the second case if $m$ is the minimum of $B$

\[
x \vee y = \begin{cases} 
  x \vee^B y & \text{if } x, y \in B \\
  x \vee^A y & \text{if } x, y \in A \text{ and } x \vee y < 1 \\
  m & \text{if } x, y \in A \text{ and } x \vee y = 1 \\
  x & \text{if } x \in B \text{ and } y \in A \\
  y & \text{if } x \in A \text{ and } y \in B;
\end{cases}
\]
If $\mathbf{S}$ is an integral residuated semilattice and $p_i(x), i = i, \ldots, n$ are residuated semilattices terms, then the equation \( \bigwedge_{i=1}^{n} p_i(x) \approx 1 \) is true in a model if and only if all the equations $p_i(x) \approx 1$ are true in the model.
A look at Jankov’s argument

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2. Since the equality relation is term definable we can encode information about $S$ into a term.

3. Any term encoding information about $S$ is called a diagram. If $S$ is finite, a **Jankov formula** for $S$ is simply any equation in the variables $X_S = \{x_s : s \in S\}$, involving diagrams of $S$. 

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4. If $S$ is a finite subdirectly irreducible residuated semilattice with monolith $\mu$ then $1/\mu$ has a minimum denoted by $\star$. 

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5. If $S = \{a_1, \ldots, a_n, \star, 1\}$ the t-diagram for $S$ is

$$T(X_S) = \wedge \{x_u \star v \leftrightarrow x_u \star x_v : u, v \in S, \star \in \{\lor, \land, \to, \leftarrow, \cdot, 1\}\}.$$  

Observe that $T(X_S)$ encodes all the operation tables of $S$ and $T(a_1, \ldots, a_n, \star, 1) = 1$ by design.
Let $S$ be a finite subdirectly irreducible integral residuated semilattice $S = \{a_1, \ldots, a_n, \ast, 1\}$ and let $B$ any integral residuated semilattice. Then $S \in IS(B)$ if and only if there are $b_1, \ldots, b_n, b_{\ast} \in B$ with $b_{\ast} \neq 1$ and

$$T(b_1, \ldots, b_n, b_{\ast}, 1) = 1.$$
The above lemma holds for integral finite subdirectly irreducible FL-algebras by redefining the t-diagram accordingly.

If \( S \) is a finite subdirectly irreducible integral residuated semilattice and \( J_T(X_S) = T(X_S) \rightarrow x \star \), then \( J_T(X_S) \approx 1 \) is a Jankov formula.

Next if \( S \in VV(B) \), then \( B \nsubseteq J_T(X_S) \approx 1 \), since \( S \) does not.

So if \( S \) is a finite subdirectly irreducible algebra, \( S \in U \) and \( W \) is the subvariety of \( U \) axiomatized by \( J_T(X_S) \approx 1 \), then \( S/ * \in W \) and \( W \subseteq W \cup S \).

Hence if \( W = W \cup S \), then \( S \) is splitting in \( U \).

This is exactly the way V. Jankov showed that any finite subdirectly irreducible Heyting algebras is splitting in any subvariety to which it belongs, but there are many cases in which \( W \neq W \cup S \).

For instance T. Kowalski and H. Ono [2000] showed that if \( U = FL_{new} \), then \( W = W \cup S \) if and only if \( S = 2 \).
1 The above lemma holds for integral finite subdirectly irreducible FL-algebras by redefining the t-diagram accordingly.

2 If S is a finite subdirectly irreducible integral residuated semilattice and $J_T(X_S) = T(X_S) \to x_*$, then $J_T(X_S) \approx 1$ is a Jankov formula.
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So if $S$ is a finite subdirectly irreducible algebra, $S \in \mathcal{U}$ and $\mathcal{W}$ is the subvariety of $\mathcal{U}$ axiomatized by $J_T(X_S) \approx 1$, then $S \not\in \mathcal{W}$ and $\mathcal{W} \subseteq \mathcal{W}^\mathcal{U}$.

Hence if $\mathcal{W} = \mathcal{W}^\mathcal{U}$, then $S$ is splitting in $\mathcal{U}$.

This is exactly the way V. Jankov showed that any finite subdirectly irreducible Heyting algebra is splitting in any subvariety of Heyting algebras to which it belongs, but there are many cases in which $\mathcal{W} \neq \mathcal{W}^\mathcal{U}$.

For instance T. Kowalski and H. Ono [2000] showed that if $\mathcal{U} = FL_{new}$, then $\mathcal{W} = \mathcal{W}^\mathcal{U}$ if and only if $S = 2$. 

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This is exactly the way V. Jankov showed that any finite subdirectly irreducible Heyting algebra is splitting in any subvariety of Heyting algebras to which it belongs, but there are many cases in which \( \mathcal{W} \neq \mathcal{W}_S^{\mathcal{U}} \).

For instance T. Kowalski and H. Ono [2000] showed that if \( \mathcal{U} = \mathcal{FL}_{ew} \), then \( \mathcal{W} = \mathcal{W}_S^{\mathcal{U}} \) if and only if \( S = 2 \).
An FL-algebra or a residuated semilattice $S$ is **divisible** if the underlying ordering is the inverse divisibility ordering, i.e. all $a, b \in S$

$$a \leq b \quad \text{if and only if} \quad \exists c, d \text{ with } a = cb = bd.$$
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Both $\mathcal{H}$ and $\mathcal{GBL}_\text{ew}$ have the finite model property.
A Wajsberg hoop is a hoop satisfying Tanaka’s equation

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Any subdirectly irreducible hoop is of the form \(F \oplus S\) where \(S\) is a totally ordered Wajsberg hoop [Blok-Ferreirim, 2000]
A Wajsberg hoop is a hoop satisfying Tanaka’s equation

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The new Jankov formula

- We want to use a Jankov-like argument but we already know that the Jankov formula $T(X_A) \rightarrow x_\star \approx 1$ is not going to be enough: we need a more refined formula.

\[
\hat{J}(X_A) = D(X_A) \rightarrow (T(X_A) \rightarrow x_\star) \land (x_\star \rightarrow x_2 \star).
\]

In this diagram we encode the fact that $\star$ is the coatom, that it is idempotent and that all the $a_i$ are distinct.

Denote $\hat{J}(X_A) = D(X_A) \rightarrow (T(X_A) \rightarrow x_\star)$; the Jankov formula we are going to use is $\hat{J}(X_A) \approx 1$. 

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$$D(X_A) = \bigwedge_{i,j=1}^n ((x_{a_i} \leftrightarrow x_{a_j}) \to x_\star) \land \bigwedge_{i=1}^n (x_{a_i} \to x_\star) \land (x_\star \to x_\star^2).$$

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- Consider the following diagram:

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In this diagram we encode the fact that $\star$ is the coatom, that it is idempotent and that all the $a_i$ are distinct.

- Define

$$\hat{J}(X_A) = D(X_A) \to (T(X_A) \to x_\star);$$

the Jankov formula we are going to use is $\hat{J}(X_A) \approx 1$. 

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Splitting residuated (semi)lattices
The rationale: the original request that $T(X_A) \rightarrow x_\star \approx 1$ was too strong, so we have to settle for less. Hence we ask that $T(X_A) \rightarrow x_\star$ be large enough to be above $D(X_A)$. 

Roughly speaking we consider the algebras in $\mathbf{GBL}$ that have the following property: if they are generated by the same number of elements as $A$ and they have a unique maximal idempotent coatom, then not all the operation tables are encoded by the t-diagram of $A$. Note that $A \not\models \hat{J}(X_A) \approx 1$ by design.
The rationale: the original request that $T(X_A) \to x_* \approx 1$ was too strong, so we have to settle for less. Hence we ask that $T(X_A) \to x_*$ be large enough to be above $D(X_A)$.

Roughly speaking we consider the algebras in $\mathcal{GBL}_{ew}$ that have the following property: if they are generated by the same number of elements as $A$ and they have a unique maximal idempotent coatom, then not all the operation tables are encoded by the t-diagram of $A$. Note that $A \not\models \widehat{J}(X_A) \approx 1$ by design.
The results

Theorem

A subdirectly irreducible algebra in $\mathcal{GBL}_{ew}$ is splitting if and only if it is $F \oplus 2$ for some finite $F \in \mathcal{GBL}_{ew}$.

Theorem

A subdirectly irreducible hoop $A$ is splitting in the variety of hoops if and only if $A = F \oplus 2$ for some finite hoop $F$. 
Advanced results

- With suitable modifications we can extend the results to many subvarieties of $\mathcal{GBL}_{ew}$ and $\mathcal{H}$ with the finite model property;
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- We can describe all the finite splitting algebras in sufficiently large normal varieties of integral GBL-algebras. (Normality is a weakening of commutativity.)
- We can specialize our results to varieties generated by totally ordered integral residuated semilattices or FL-algebras; such varieties are called representable and they have several properties that can be used to facilitate our investigation.
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- They are both known to have the finite model property [?] and their splitting algebras can be completely characterized.
- We can also try to dispose of divisibility altogether, and this is the subject of our ongoing research.