Evolution of Algebraic Terms 3

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Algebras and Lattices in Hawaii
Honolulu, HI

Problem: *Given a finite algebra* $\mathbf{G}$ *and a* k*-ary operation* $t : G^k \to G$ *that is known to be a term operation of* $\mathbf{G}$, *can we find a computationally efficient method to produce a term whose term operation is* t*?*

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G is **term continuous** TC if its term to term operation is (relative to appropriate metrics) continuous.

1. Evolution of algebraic terms 1: term to term operation continuity, *International Journal of Algebra and Computation*, Vol. 23, No. 5 (2013) 1175-1205.

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2. Evolution of algebraic terms 2: the deep drilling algorithm, (with L. Spector and M. Keijzer), *International Journal of Algebra and Computation*, Vol. 26, No. 6 (2016) 1141- 1176.

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2. Evolution of algebraic terms 2: the deep drilling algorithm, (with L. Spector and M. Keijzer), *International Journal of Algebra and Computation*, Vol. 26, No. 6 (2016) 1141- 1176.

3. Evolution of algebraic terms 3: beam algorithms and term continuity, (with L. Spector), *International Journal of Algebra and Computation*, 32pp (in press).

A k-ary **partial term** is a term $f(\vec{x}, \Diamond)$ in variables $x_0, x_1, \ldots, x_{k-1}$ that contains exactly one occurrence of a new variable $\Diamond$, e.g.,

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$f(\vec{x}, \Diamond)$ is **valid with respect to** t if there is an operation $h : G^k \to G$ such that

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Theorem. *If* $\mathbf{G}$ *is idemprimal, then there is a term* $u(\vec{x})$ *such that* $f(\vec{x}, u(\vec{x}))^{\mathbf{G}} = t$ *if and only if* $f(\vec{x}, \Diamond)$ *is valid with respect to* t.

A Beam Enumeration Algorithm

A Beam Enumeration Algorithm

G - idemprimal, $t : G^k \to G$, finite set M of test terms

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$\lozenge$ t

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$\Diamond \; t$

Generate $\Diamond u(\vec{x})$ or $u(\vec{x})\Diamond$ valid w.r.t. t

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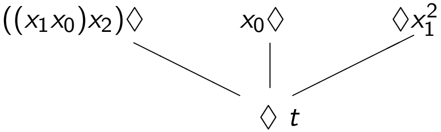

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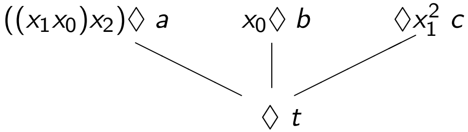

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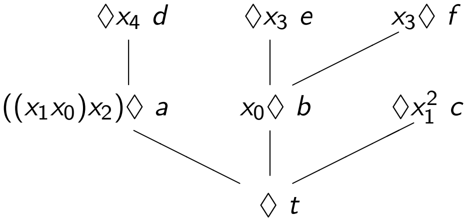

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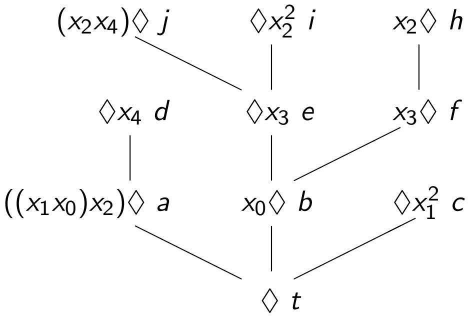

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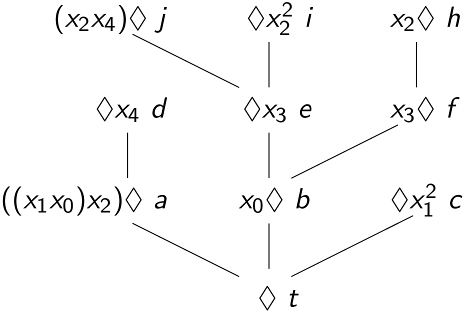

Assume $u(\vec{x}) \in M$ and $u(\vec{x})^{\mathbf{G}} = j$.

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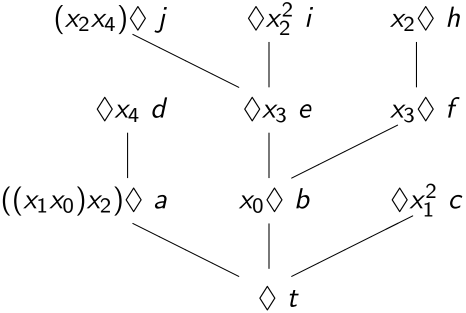

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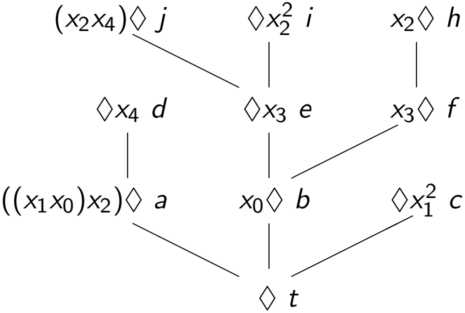

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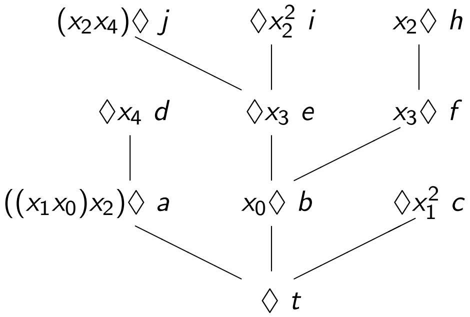

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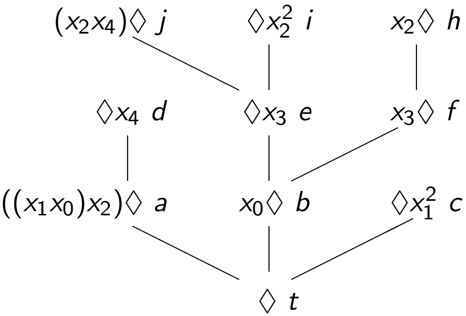

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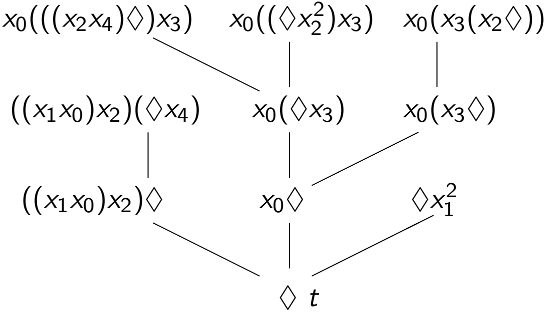

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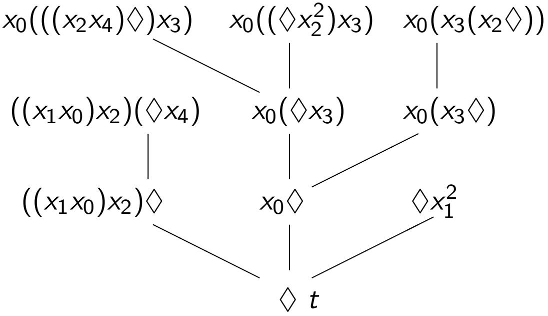

A partial term $f(\vec{x}, \Diamond)$ is **constant** on **G** if all terms $u(\vec{x})$ produce the same term operation $f(\vec{x}, u(\vec{x}))^{\mathbf{G}}$.

Idemprimality Theorem. *If* **G** *is* **idemprimal** (IPr) *and a BEA produces a constant partial term, then the BEA will terminate with a representation of the target operation.*

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and the proportion of partial terms of height at most H *that are constant converges to* λ *as* $H \to \infty$*.*

$\mathbf{G} := \langle\{0, 1, 2, 3\}, *\rangle$

$*$	0	1	2	3
0	1	3	1	2
1	1	2	2	3
2	2	0	0	3
3	2	2	1	2

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Example: Find a discriminator term for **G**.

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$d(x, y, z) =$

$$
(((((xy)x)x)(((xy)x)((zx)(yx))))(((xy)(((xx)x)x))(((((yx)(yx))x)((yx)y))(((yx)((xy)(y(yx))))((z(z((xz)
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492 variable occurrences,

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\begin{aligned}
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&((xx)((xx)((x(x((xx)(zz))))((zx)((xx)(((((y((yy)y))y)((y(y(xz)))((((yz)((zz)z))((z(xy))z))(((zz)(z((yx)\\
&(zy))))(((x(y(zx)))(((yx)(zx))(zx)))(((((zx)y)x)(yz))(((zz)(((((yz)x)((zy)x))(((yx)(yx))((zz)x)))\\
&((x((y(yy))x))(((xx)(((xy)y)(zx)))((((((xz)y)z)(yy))(yz))(y(x(xy))))((y((zy)(yz)))(((xz)(zx))\\
&(y(yx)))))))))(xy))))))))(yy(y((y(yy))y))\\
&)))))))))))))))))))))))))))))))))))))
\end{aligned}
$$

492 variable occurrences, 11.3 seconds.

Thanks for listening!

— DC