# A tale of three papers: a beautiful theorem, four open problems, a surprising connection

My joint work with Bill, JB and Ralph

Brian A. Davey
Algebras and Lattices in Hawai'i
May 23, 2018

# Outline

- A survival guide to natural dualities
- Paper 1: Davey, Nation, McKenzie and Pálfy (1994)
  A beautiful theorem
- Paper 2: Davey, Idziak, Lampe, McNulty (2000)
  Four open problems
- Paper 3: Clark, Davey, Freese, Jackson (2004)
  A surprising connection
- Bonus: A further surprising connection (2018)
  Lifting full dualities from the finite level

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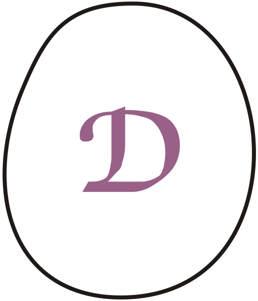
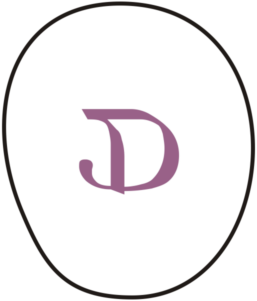
# Priestley duality



Bounded distributive lattices



Priestley spaces

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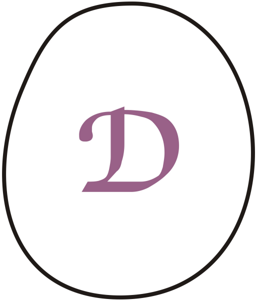

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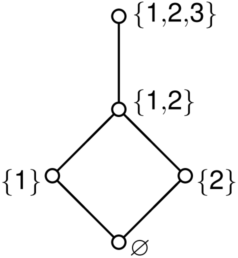


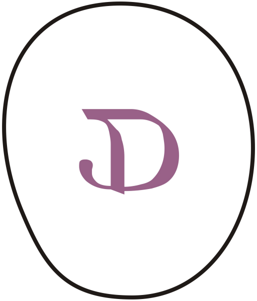

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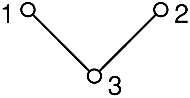

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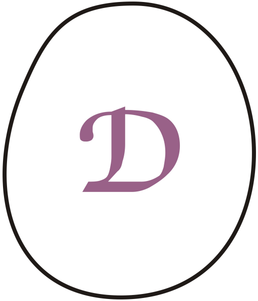

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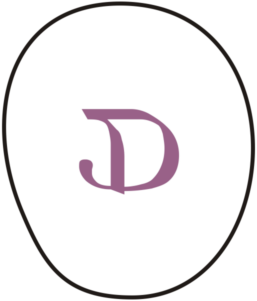
?



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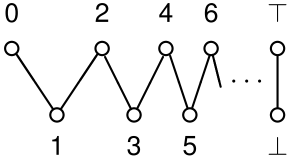

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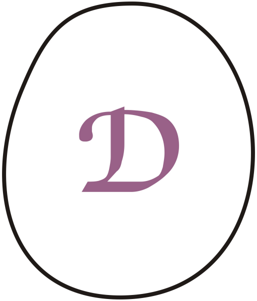

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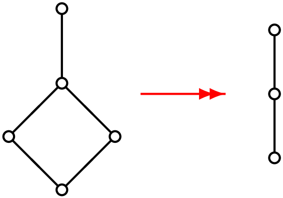

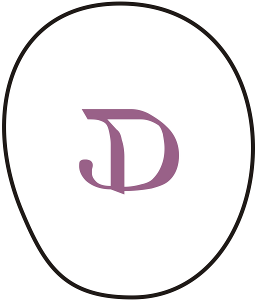

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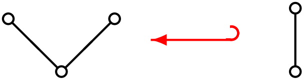

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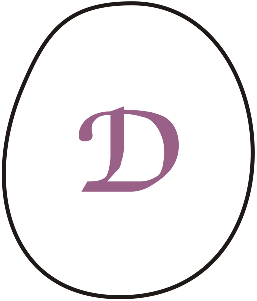
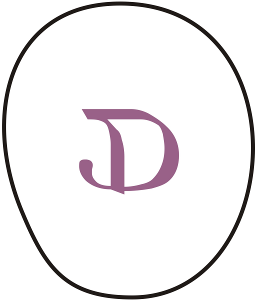


Bounded distributive lattices

$\mathfrak{D} = \mathrm{ISP}(\mathbf{D})$, where

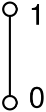
$\mathbf{D} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$

Priestley spaces

$\mathrm{IS_cP^+}(\mathbb{D})$, where

$\mathbb{D} = \langle \{0, 1\}; \leqslant, \mathfrak{T} \rangle$



# Natural dualities: alter egos

Generalizing this and many other examples, we start with a finite algebra **M** and wish to find a dual category for the quasivariety $\mathcal{A} := \mathsf{ISP}(\mathbf{M})$.

## An alter ego of a finite algebra

A finitary structure $\mathbb{M} = \langle M; G, H, R, \mathcal{T}\rangle$ is an alter ego of **M** if

- $G$ is a set of compatible operations on **M**,
- $H$ is a set of compatible partial operations on **M**,
- $R$ is a set of compatible relations on **M**,
- $\mathcal{T}$ is the discrete topology on $M$.

An alter ego of **M** is often denoted by $\underset{\sim}{\mathbf{M}}$ instead of $\mathbb{M}$.

# Natural dualities: categories and functors

Fix a finite algebra **M** and let $\mathbb{M} = \langle M; G, H, R, \mathcal{T} \rangle$ be an alter ego of **M**.

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## The categories $\mathcal{A}$ and $\mathcal{X}$

- Define $\mathcal{A} := \mathrm{ISP}(\mathbf{M})$: the algebraic category of interest.
- Define $\mathcal{X} := \mathrm{IS_cP^+}(\mathbb{M})$: the potential dual category for $\mathcal{A}$.

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## The contravariant functors $D$ and $E$

- There are natural hom-functors $D\colon \mathcal{A} \to \mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{A}$.
- For each algebra **A** in $\mathcal{A}$,

$$D(\mathbf{A}) = \hom(\mathbf{A}, \mathbf{M}) \leqslant \mathbb{M}^{A}.$$

- For each structure **X** in $\mathcal{X}$,

$$E(\mathbf{X}) = \hom(\mathbf{X}, \mathbb{M}) \leqslant \mathbf{M}^{X}.$$

# Natural dualities: embeddings

## Natural embeddings

For every $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathfrak{X}$, there are naturally defined embeddings

$$e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A}) \qquad \text{and} \qquad \varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X}).$$

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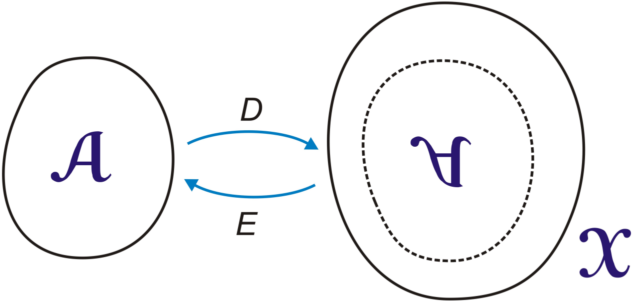
These embeddings yield natural transformations

$$e \colon \mathrm{id}_{\mathcal{A}} \to ED \qquad \text{and} \qquad \varepsilon \colon \mathrm{id}_{\mathcal{X}} \to DE,$$

and $\langle D, E, e, \varepsilon \rangle$ is a dual adjunction between $\mathcal{A}$ and $\mathcal{X}$.

# Duality

If $e_{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism, for all $\mathbf{A}$ in $\mathcal{A}$, then we say that $\mathbb{M}$ yields a duality on $\mathcal{A}$ (or that $\mathbb{M}$ dualises $\mathbf{M}$).



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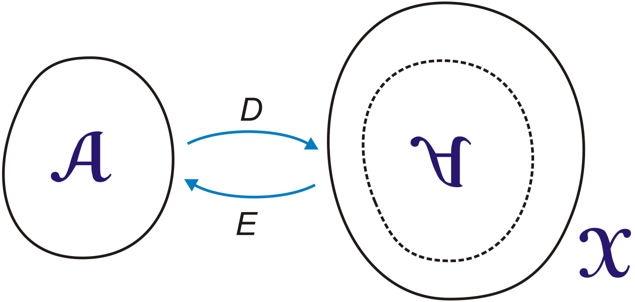
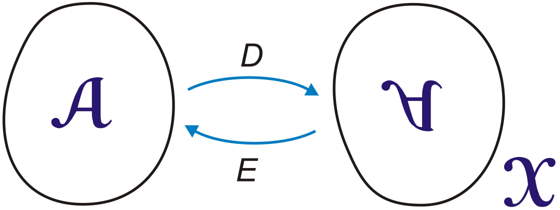



If some alter ego dualises $\mathbf{M}$, then we say that $\mathbf{M}$ is dualisable.

# Full duality

If, in addition, $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X})$ is an isomorphism, for all $\mathbf{X}$ in $\mathcal{X}$, then we say that $\mathbb{M}$ yields a full duality on $\mathcal{A}$ (or that $\mathbb{M}$ fully dualises $\mathbf{M}$).



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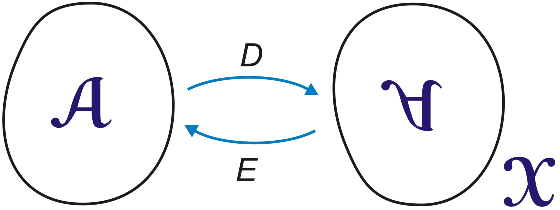


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- The two-element implication algebra $\mathbf{I} := \langle \{0, 1\}; \to \rangle$ is not dualisable.

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# Examples

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- It was this final example that led to Paper 1 co-authored with JB.

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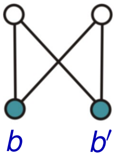
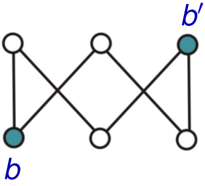
# Distance in ordered sets – crowns

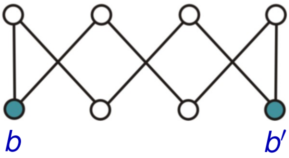
$\mathbf{C}_2$




$\mathbf{C}_3$




$\mathbf{C}_4$




$d(b, b') = 2$

$d(b, b') = 3$

$d(b, b') = 4$

# Antipodes


# The definition of a braid

Let **P** be a connected ordered set with at least three elements.

- Define the distance function $d\colon P^2 \to \mathbb{N}$ by

  $$d(a, b) = \text{number of edges in a min-sized fence from } a \text{ to } b.$$

- The reach of **P** is the 'maximum' distance between elements:

  $$r(\mathbf{P}) := \sup\{\, d(a, b) \mid a, b \in P \,\}.$$

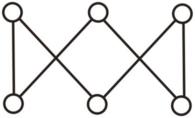
- Two elements $b, b'$ of **P** are antipodal if $d(b, b') = r(\mathbf{P})$.

- The ordered set **P** is a braid if every element of $P$ has a unique antipodal element in **P**.

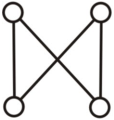
# Examples of braids

0. The crown $\mathbf{C}_n$ is a braid of reach $n$.

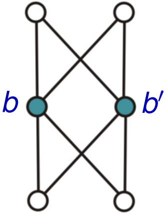
$\mathbf{C}_3$



1. For a chain $\mathbf{C}$, define the tower $\mathbf{T}_\mathbf{C}$ to be the linear sum over $\mathbf{C}$ of 2-element antichains.



$\mathbf{T}_2 = \mathbf{C}_2$

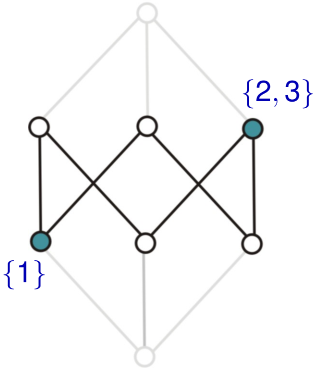


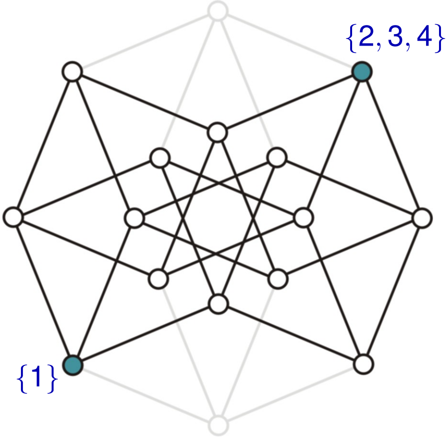

$\mathbf{T}_3$

The towers $\mathbf{T}_\mathbf{C}$ are the only braids of reach 2.

# Examples of braids, continued

2. Let $S$ be a set with $|S| \geqslant 3$ and define $P := \wp(S) \setminus \{\varnothing, S\}$.



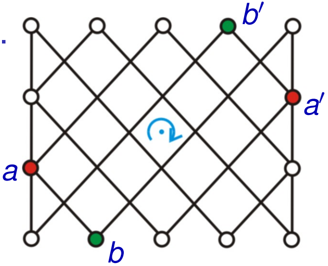




Then $\langle P; \subseteq\rangle$ is a braid of reach 3, where $a' = S \setminus a$.

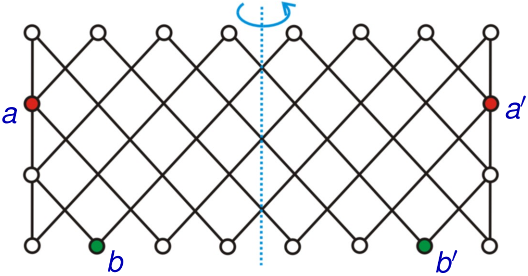
# Examples of braids – cyclone (= chain link) fences


# Examples of braids – cyclone fences

3.




$\mathbf{F}_{3,3}$




$\mathbf{F}_{3,4}$

$\mathbf{F}_{h,r}$ has height $h$, reach $r \geqslant 3$, and width $h(r-2)+2$.

# The pathological behaviour of braids

Braids-are-Pathological Theorem

*Let* **P** *be a finite braid and let* $f\colon P^n \to P$ *be order-preserving.*

*If* $f$ *is idempotent,*

$$(\text{i.e., } f(a, a, \ldots, a) = a, \text{ for all } a \in P)$$

*then* $f$ *is a projection.*

We say that **P** is idempotent trivial.

# Why does this make braids pathological?

## Examples

Familiar algebras have interesting idempotent operations.

- Groups, rings, vector spaces

  Define $f\colon G^3 \to G$ by $f(x, y, z) = x - y + z$.

  Then $f$ is idempotent, since $f(x, x, x) = x$, for all $x \in G$.

- Boolean algebras, lattices, semilattices

  Define $f\colon A^2 \to A$ by $f(x, y) = x \vee y$.

## Mal'cev conditions

- Many important Mal'cev conditions involve idempotent operations; for example, congruence modularity.

# Abelian groups of exponent 2

Let $\mathbf{A} = \langle A; + \rangle$ be a non-trivial finite abelian group satisfying $x + x = 0$, i.e., a group of the form $(\mathbb{Z}_2)^m$.

The polynomial operations of $\mathbf{A}$ are the maps $f\colon A^n \to A$, for some $n \geqslant 1$, given by

$$f(x_1, \ldots, x_n) := x_{i_1} + \cdots + x_{i_t} + c, \quad \text{for some } c \text{ from } A.$$

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The collection $\mathcal{A}$ of all polynomial operations of $\mathbf{A}$

(a) is a clone, i.e.,
- contains all the projections $\pi_i\colon A^n \to A$,
- is closed under composition,

(b) contains all the constant operations $c\colon A^n \to A$,

(c) is 2-idempotent trivial, i.e, the only polynomials $f\colon A^2 \to A$ such that $f(a, a) = a$, for all $a \in A$, are the two projections;

(d) is not idempotent trivial, i.e, there is some $f\colon A^n \to A$ such that $f(a, a, \ldots, a) = a$, for all $a \in A$, and $f$ is not a projection – take $n = 3$ and $f(x_1, x_2, x_3) := x_1 + x_2 + x_3$.

# A beautiful theorem

## Abelian Group Theorem

*Let $\mathcal{C}$ be a collection of operations on a finite set $A$.*

*Assume that $\mathcal{C}$*

(a) *is a clone,*

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*Then there is a binary operation $+$ in $\mathcal{C}$ such that*

- $\langle A; + \rangle$ *is an abelian group satisfying $x + x = 0$, and*
- $\mathcal{C}$ *is the collection of polynomial operations of $\langle A; + \rangle$.*

*In particular, $|A| = 2^m$, for some $m \geqslant 1$.*

# Abelian groups applied to braids

## Corollary 1

*Let* **P** *be a finite ordered set and assume that every idempotent order-preserving function* $f\colon P^2 \to P$ *is a projection. Then every idempotent order-preserving function* $f\colon P^n \to P$ *is a projection, for all* $n \geqslant 2$.

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## Proof.

Suppose that **P** is 2-idempotent trivial but not idempotent trivial. Let $\mathcal{C}$ be the clone of order-preserving functions on **P**.

- Then $\mathcal{C}$ satisfies conditions (a)–(e) of the theorem.
- Thus there is a binary operation $+$ in $\mathcal{C}$ such that $\langle A; +\rangle$ is an abelian group satisfying $x + x = 0$.
- This implies that **P** is an antichain and so is not 2-idempotent trivial, $\lightning$. □

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## Braids-are-Pathological Theorem

*Every finite braid is 2-idempotent trivial and therefore idempotent trivial.*

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## Braids-are-Pathological Theorem

*Every finite braid is* 2*-idempotent trivial and therefore idempotent trivial.*

## Corollary 2

*If Arrow's Theorem is true when there are only* 2 *voters, then it it is true for any number* $n$ *of voters, with* $n \geqslant 2$.

# Outline

- A survival guide to natural dualities
- Paper 1: Davey, Nation, McKenzie and Pálfy (1994)
  A beautiful theorem
- Paper 2: Davey, Idziak, Lampe, McNulty (2000)
  Four open problems
- Paper 3: Clark, Davey, Freese, Jackson (2004)
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  Lifting full dualities from the finite level

# Open problems

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- Finiteness is necessary: the flat semilattice $\mathbf{F} := \langle \mathbb{Z} \cup \{\bot\}; \wedge, s, s^{-1}, \bot \rangle$ is self-dualising and generates a non-finitely based variety.

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In 1976, Park conjectured that every finite algebra of finite signature that generates a residually finite variety must be finitely based.

## Open Problem 2

Is every finite dualisable algebra of finite signature that generates a residually finite variety necessarily finitely based?

# Graph algebras

Let $\mathbf{G} = \langle V; r \rangle$ be a graph, i.e., $r$ is a symmetric binary relation on the set $V$. The graph algebra of $\mathbf{G}$ is the algebra $\mathbf{A}(\mathbf{G}) := \langle V \mathbin{\dot\cup} \{0\}; \cdot \rangle$ where

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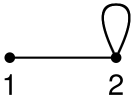
Example

**M**




**A**(**M**)

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 2 | 0 | 2 | 2 |

The groupoid **A**(**M**) was invented by V. L. Murskiĭ in 1965.

# Graph algebras

**Theorem**

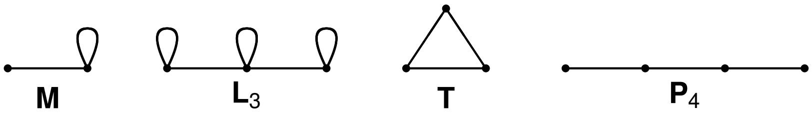
*The following statements are equivalent for any finite graph* $\mathbf{G}$*.*

(i) $\mathbf{A}(\mathbf{G})$ *is dualisable.*

(ii) *Each connected component of* $\mathbf{G}$ *is either complete* (*with all loops*)*, bipartite complete* (*with no loops*)*, or a loose vertex.*

(iii) $\mathbf{A}(\mathbf{G})$ *is finitely based.*

# Graph algebras

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*The following statements are equivalent for any finite graph* **G***.*

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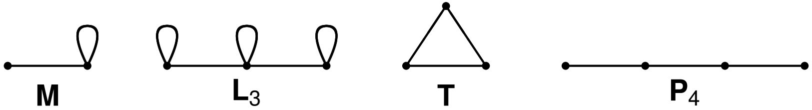


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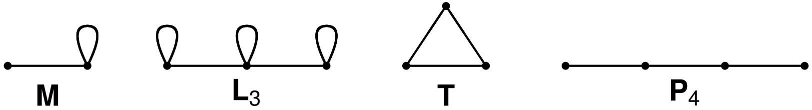


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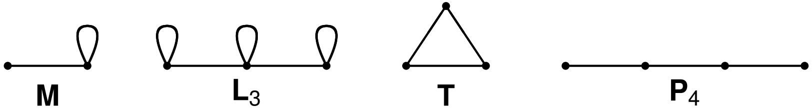


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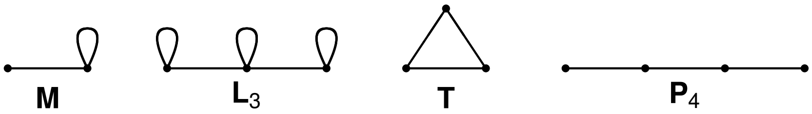


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# Inherent non-dualisability

- The important concept of inherent non-dualisability was introduced in this paper.

### Definition

A finite algebra $\mathbf{M}$ is inherently non-dualisable if every finite algebra $\mathbf{M}'$ that has $\mathbf{M}$ as a subalgebra is non-dualisable.

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- The very useful Non-Dualisability Lemma and Inherent Non-Dualisability Lemma also come from this paper.

## The ND Lemma and the IND Lemma

Every known example of non-dualisability and inherent non-dualisability can be proved by applying these lemmas.

# $\kappa$-dualisability

There is no need to insist that the signature of an alter ego $\mathbb{M}$ be finitary.

## Definition

A finite algebra **M** is $\boldsymbol{\kappa}$-dualisable if $\mathbb{M}_{\boldsymbol{\kappa}} := \langle M; R_{\boldsymbol{\kappa}}, \mathcal{T}\rangle$ yields a duality on ISP(**M**), where $R_{\boldsymbol{\kappa}}$ is the set of all **less-than-$\kappa$-ary** compatible relations on **M**.

- Hence dualisability in the usual sense is precisely $\omega$-dualisability.

## Open Problem 3

Is there a finite algebra that is $\boldsymbol{\kappa}$-dualisable, for some cardinal $\boldsymbol{\kappa}$, but is not $\boldsymbol{\omega}$-dualisable?

# The Hanf number for dualisability

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To each finite algebra **A** assign the smallest $\kappa$ such that **A** is $\kappa$-dualisable, if such $\kappa$ exists, and $\infty$ otherwise. The resulting set of cardinals (plus $\infty$) is the dualisability spectrum $\mathrm{S_d}$.

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# The Hanf number for dualisability

### Open Problem 4

Find the Hanf number for dualisability.

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## Open Problem 4

Find the Hanf number for dualisability.

## What is known?

- Pitkethly (2011) proved that the Hanf number for dualisability is at least $\aleph_2$.
- Pitkethly (2010) proved that the Hanf number for the class of unary algebras is either $\omega$ or at least $\aleph_2$.

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# Standardness

- For clarity, given a finite structure $\mathbb{M} = \langle M; G, H, R \rangle$, we shall define
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- The search for such descriptions led to the concept of standardness.

# uH sentences

A universal Horn sentence (uH sentence, for short) in the language $(G, H, R)$ is a universally quantified formula of the form

$$\gamma(\vec{v}), \quad \bigvee_{i=1}^{k} \neg\alpha_i(\vec{v}), \quad \text{or} \quad \left(\bigwedge_{i=1}^{k} \alpha_i(\vec{v})\right) \to \gamma(\vec{v}),$$

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- We say that the structure $\mathbb{M}$ (and that the corresponding topological structure $\mathbb{M}_{\mathcal{T}}$) is standard if

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## Examples

- The cyclic group $\mathbb{C}_n = \langle C_n; \cdot, ^{-1}, 1 \rangle$ is standard:
  - $\mathsf{ISP}^+(\mathbb{C}_n)$ is the class of abelian groups satisfying $x^n = 1$, and
  - $\mathsf{IS_cP^+}((\mathbb{C}_n)_{\mathcal{T}})$ is the class of Boolean topological abelian groups satisfying $x^n = 1$.
- The two-element chain $\mathbb{2} = \langle \{0, 1\}; \leqslant \rangle$ is not standard:
  - $\mathsf{ISP}^+(\mathbb{2})$ is the class of ordered sets, while
  - $\mathsf{IS_cP^+}(\mathbb{2}_{\mathcal{T}})$ is the class of Priestley spaces, which is not the class of Boolean topological ordered sets (Stralka 80).

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- In Paper 3, co-authored with Ralph, we found a surprising connection between two purely algebraic conditions on a finite algebra and the topological condition of standardness.

# Finitely Determined Syntactic Congruences (FDSC)

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- A class $\mathcal{K}$ of algebras has Finitely Determined Syntactic Congruences if there is a finite subset $F$ of $T_x$ such that $\mathrm{Syn}(\theta) := \theta_F$, for every equivalence relation $\theta$ on every algebra in $\mathcal{K}$.

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- A class $\mathcal{K}$ of algebras has Term Finite Principal Congruences if there is a finite subset $F$ of $T_x$ such that $C_F^{\mathbf{A}}(a, b) = \mathrm{Cg}^{\mathbf{A}}(a, b)$, for all $a, b$ in every algebra **A** in $\mathcal{K}$.

# FDSC = TFPC

## Theorem

- *Let* **A** *be an algebra and let* $F \subseteq T_X$*. Then* $F$ *determines syntactic congruences on* **A** *if and only if* $F$ *determines principal congruences on* **A**.

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# FDSC = TFPC

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- *Let **A** be an algebra and let $F \subseteq T_X$. Then $F$ determines syntactic congruences on **A** if and only if $F$ determines principal congruences on **A**.*
- *A class $\mathcal{K}$ of algebras has FDSC if and only if it has TFPC.*
- *If a variety has DPC, then it has TFPC and therefore has FDSC.*

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## Algebras with FDSC (= TFPC)

- Monoids: $F = \{z_1 x z_2\}$.
- Every finitely generated variety of unary algebras.
- Groups: $F = \{z_1 x z_2, z_1 x^{-1} z_2\}$.
- Semigroups: $F = \{x, z_1 x, x z_2, z_1 x z_2\}$
- Every finitely generated variety of lattices.

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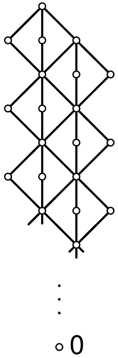
- The variety of modular lattices, and therefore the variety of lattices, does not have FDSC.



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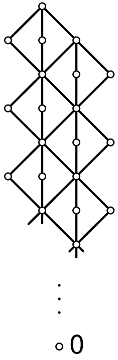
- McKenzie's algebra $\mathbf{A}(\mathcal{T})$ generates a variety without FDSC in the case that the Turing machine $\mathcal{T}$ does not halt.
  (The algebra $\mathbf{Q}_\omega$ constructed by McKenzie is SI and has a compatible Boolean topology.)



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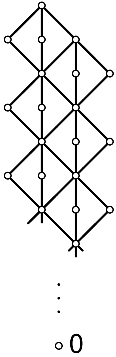


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(The algebra $\mathbf{Q}_\omega$ constructed by McKenzie is SI and has a compatible Boolean topology.)
- In a recent preprint, Nurakunov, Stronkovski (2018) prove that it is undecidable whether a finite algebra generates a variety with FDSC. They use using the algebra $\mathbf{A}'(\mathcal{T})$ constructed by Moore (2015).

# The surprising connection

### FDSC-HSP Theorem

*Let* $\mathbb{M} = \langle M; G \rangle$ *be a finite algebra. Assume that*

- $\mathsf{HSP}(\mathbb{M})$ *has FDSC, and*
- $\mathsf{HSP}(\mathbb{M}) = \mathsf{ISP}(\mathbb{M})$.

*Then* $\mathbb{M}$ *is standard and hence* $\mathsf{IS_cP^+}(\mathbb{M}_{\mathcal{T}}) = \mathsf{Mod_{BT}}(\mathsf{Th_{uH}}(\mathbb{M}))$.

# Outline

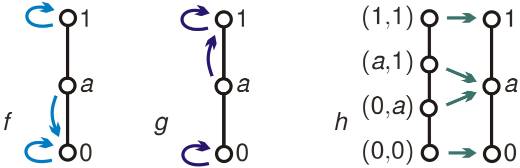
- A survival guide to natural dualities
- Paper 1: Davey, Nation, McKenzie and Pálfy (1994)
  A beautiful theorem
- Paper 2: Davey, Idziak, Lampe, McNulty (2000)
  Four open problems
- Paper 3: Clark, Davey, Freese, Jackson (2004)
  A surprising connection
- Bonus: A further surprising connection (2018)
  Lifting full dualities from the finite level

# Lifting full dualities from the finite level

- Let **M** be a finite algebra and let $\mathbb{M} = \langle M; G, R, \mathcal{T} \rangle$ be an alter ego of **M** with $G \cup R$ finite (and $H = \varnothing$).

  If $\mathbb{M}$ yields a full duality between $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$, then $\mathbb{M}$ yields a full duality between $\mathcal{A}$ and $\mathcal{X}$. (Hofmann 02)

- Let $\mathbf{3} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$ be the three-element chain and let $\mathbb{3} := \langle \{0, a, 1\}; f, g, h, \mathcal{T} \rangle$, where




- The alter ego $\mathbb{3}$
  - yields a duality between $\mathcal{D} = \mathsf{ISP}(\mathbf{3})$ and $\mathcal{X} := \mathsf{IS_cP^+}(\mathbb{3})$,
  - yields a full duality between $\mathcal{D}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$,
  - but does not yield a full duality between $\mathcal{D}$ and $\mathcal{X}$.

(Davey, Haviar and Willard 05)

# A further surprising connection

## Full dualities and standardness

Davey, Pitkethly, Willard (2018) found a surprising connection between full dualities and standardness.

### Theorem

*Let* $\mathbf{M}$ *be a finite algebra, let* $\mathcal{A} = \mathsf{ISP}(\mathbf{M})$ *and let* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *be alter egos of* $\mathbf{M}$*. Assume that*

- $\mathbb{M}_1$ *yields a full duality between* $\mathcal{A}$ *and* $\mathcal{X}_1$*,*
- $\mathbb{M}_1$ *is standard, and*
- $\mathbb{M}_2$ *yields a full duality between* $\mathcal{A}_{\mathrm{fin}}$ *and* $(\mathcal{X}_2)_{\mathrm{fin}}$*.*

*Then* $\mathbb{M}_2$ *yields a full duality between* $\mathcal{A}$ *and* $\mathcal{X}_2$ *and* $\mathbb{M}_2$ *is standard.*

### Quasi-primal algebras

Every alter ego of a quasi-primal algebra that yields a full duality at the finite level yields a full duality and is standard.