SOLVING EQUATIONS – KITH AND KIN

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Algebras and Lattices in Hawaii
in honor of Ralph Freese, Wiliam Lampe, J.B. Nation

Solving equations

- Linear equations
- Diophantine equations – Hilbert's 10th problem
- SAT

- POLSAT – equations of polynomials over finite algebras
- SYSPOLSAT – finite systems of equations of polynomials over finite algebras

Equations satisfiability and CSP

Fact (Feder, Madelaine & Stewart 2004; Larose & Zádori 2006)

- for every finite relational structure $\mathbb{D}$ there is a finite algebra $\mathbf{A}[\mathbb{D}]$ such that the problem $\mathrm{CSP}(\mathbb{D})$ is polynomially equivalent to $\textsc{SysPolSat}(\mathbf{A}[\mathbb{D}])$,
- for every finite algebra $\mathbf{A}$ there exists a relational structure $\mathbb{D}[\mathbf{A}]$ such that the problems $\textsc{SysPolSat}(\mathbf{A})$ and $\mathrm{CSP}(\mathbb{D}[\mathbf{A}])$ are polynomially equivalent.

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- for every finite relational structure $\mathbb{D}$ there is a finite algebra $\mathbf{A}[\mathbb{D}]$ such that the problem $\mathrm{CSP}(\mathbb{D})$ is polynomially equivalent to $\textsc{PolSat}(\mathbf{A}[\mathbb{D}])$.
- TBD ??

Defining equations satisfiability?

- fixed finite algebra as a template

Examples

Groups (Goldmann & Russell 1999)

Polynomial satisfiability problem ($\textsc{PolSat}$) is NP-complete for non-solvable groups and in P for nilpotent groups.

Rings (Burris & Lawrence 1993; Horváth 2011)

Let $\mathbf{A}$ be a finite ring. Then $\textsc{PolSat}(\mathbf{A})$ is in P, whenever $\mathbf{A}$ is nilpotent and NP-complete otherwise.

Lattices (Schwarz 2004)

Let $\mathbf{A}$ be a finite lattice. Then $\textsc{PolSat}(\mathbf{A}) \in \mathrm{P}$ if $\mathbf{A}$ is distributive and NP-complete otherwise.

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Kosicka Bela observations 2003

For (solvable but non-nilpotent) symmetric group S_3:

- PolSat$(\mathrm{S}_3; \cdot, {}^{-1})$ is in P (Horváth & Szabó)
- PolSat$(\mathrm{S}_3; \cdot, {}^{-1}, \text{a couple of additional polynomials})$ is NP-complete.

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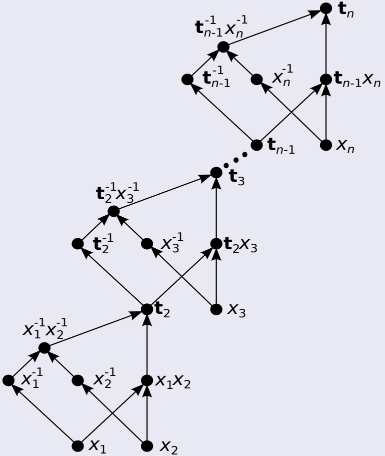
Fact (Horváth & Szabó 2012)

For (solvable but non-nilpotent) alternating group A_4:

- $\textsc{PolSat}\big(\mathrm{A}_4; \cdot, {}^{-1}\big)$ is in P,
- $\textsc{PolSat}\big(\mathrm{A}_4; \cdot, {}^{-1}, [,]\big)$, where $[x, y] = x^{-1}y^{-1}xy$, is NP-complete.

exponential syntactic tree vs polynomial size circuit

$$t_n(x_1, x_2, \ldots, x_n) = [\ldots[[x_1, x_2], x_3]\ldots x_n]$$



Definition

$\textsc{Csat}(\mathbf{A})$
given a circuit over $\mathbf{A}$ with two output gates g_1, g_2
is there a valuation of input gates $\overline{x} = (x_1, \ldots, x_n)$ that gives the same output on g_1, g_2, i.e. $g_1(\overline{x}) = g_2(\overline{x})$.

Back to groups

PolSat (Goldmann & Russell 1999)

Polynomial satisfiability problem (PolSat) is NP-complete for non-solvable groups and in P for nilpotent groups.

CSat (Horváth & Szabó 2011)

Circuit satisfiability problem (CSat) is NP-complete for non-nilpotent groups and in P for nilpotent groups.

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- quotients

Quotients

Fact

There is a finite algebra $\mathbf{A}$ with a congruence α such that $\textsc{Csat}(\mathbf{A})$ is in P while $\textsc{Csat}(\mathbf{A}/\alpha)$ is NP-complete.

Fact (Klíma, Tesson & Thérien 2007)

There is a finite algebra $\mathbf{A}$ with a congruence α such that $\mathrm{S}\textsc{Csat}(\mathbf{A})$ is in P while $\mathrm{S}\textsc{Csat}(\mathbf{A}/\alpha)$ is NP-complete.

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Theorem (LICS'18)

Let $\mathbf{A}$ be a finite algebra of finite type from a congruence modular variety.

1. If $\mathbf{A}$ has no quotient $\mathbf{A}'$ with $\textsc{CSat}(\mathbf{A}')$ being NP-complete then $\mathbf{A}$ is isomorphic to a direct product $\mathbf{N} \times \mathbf{D}$, where $\mathbf{N}$ is a nilpotent algebra and $\mathbf{D}$ is a subdirect product of 2-element algebras each of which is polynomially equivalent to the 2-element lattice.
2. If $\mathbf{A}$ decomposes into a direct product $\mathbf{N} \times \mathbf{D}$, where $\mathbf{N}$ is a supernilpotent algebra and $\mathbf{D}$ is a subdirect product of 2-element algebras each of which is polynomially equivalent to the 2-element lattice, then for every quotient $\mathbf{A}'$ of $\mathbf{A}$ the problem $\textsc{CSat}(\mathbf{A}')$ is solvable in polynomial time.

easy, moderate and sometimes heavy use of TCT

Poly-time algorithms

If $\mathbf{A}$ is a supernilpotent algebra (or a distributive lattice) then there is a constant d so that for each natural number n there is $S_n \subseteq A^n$ such that

- $|S_n|$ is $O(n^d)$,
- for two n-ary polynomials s and t the equation $s(\overline{x}) = t(\overline{x})$ has a solution $\overline{x} \in A^n$ iff it has a solution in S_n.

$\sim$ & $\sim$ & Kawałek

There exist nilpotent (but not supernilpotent) algebras $\mathbf{A}$ such that:

- $\textsc{Csat}(\mathbf{A})$ is in P,
- $\textsc{Csat}(\mathbf{A})$ can not be solved in polynomial time using algorithm checking a small set of potential solutions which depends only on the number of input gates of a given circuit (unless $\mathrm{P} = \mathrm{NP}$).

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$\mathbf{A} = (\mathbb{Z}_6; +, f)$, where $f(x) = x \mod 2$.

	tractable	open	intractable
CEQV	supernilpotent Aichinger & Mudrinski	nil but not supernil	non nilpotent
CSAT	supernil $\times$ DL-like	nil but not supernil	non (nil $\times$ DL-like)
MCSAT	affine $\times$ DL-like	—	otherwise
SCSAT	affine Gaussian elimination	—	otherwise Larose & Zádori