Is Supernilpotence Super Nilpotence?

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Question for today: Is the word "supernilpotent" a good choice for this concept?

Silly answer: Yes.

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Here, the implication goes the other way: Nilpotence implies supernilpotence, but the converse fails.

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Interestingly, Peirce also originated *idempotent* ("having same power"), but he meant "$\exists n \geq 2(A^n = A)$".

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Capelli's paper appeared in 1884, 12 years before the group commutator was introduced.

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However, (Bruck's students) G. Glauberman and C. Wright proved that centrally nilpotent Moufang loops are products of prime power order loops, and that prime power order Moufang loops are centrally nilpotent.

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The (β-)twin monoid of $\mathbf{A}$ is the submonoid of $\mathrm{Pol}_1(\mathbf{A})$ consisting of (β-)twins of the identity function.

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Note. Two groups defined on the same set and having the same clone can have nonisomorphic multiplication groups and nonisomorphic translation groups, but must have isomorphic twin monoids.

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(1)⇒(2): Vaughan-Lee, Freese-McKenzie.

(2)⇒(3): Blok-Berman.

(3)⇒(4)⇒(1): Kearnes.

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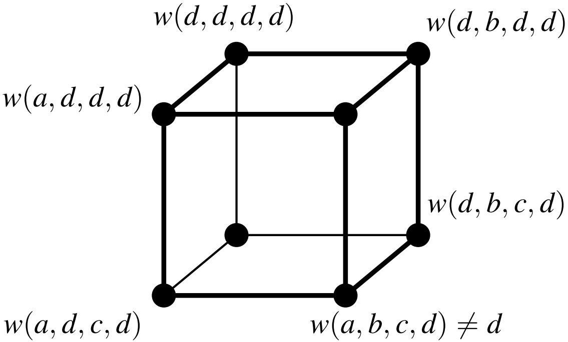

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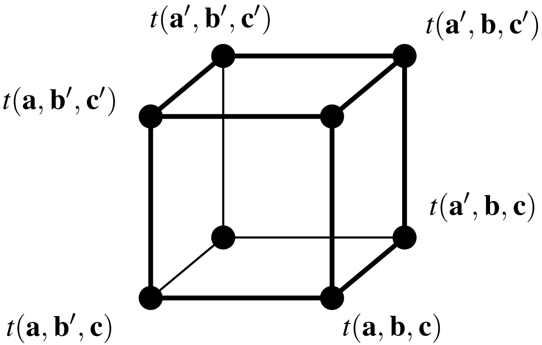

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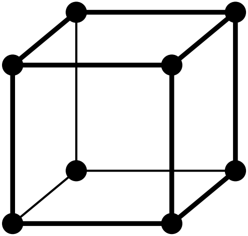

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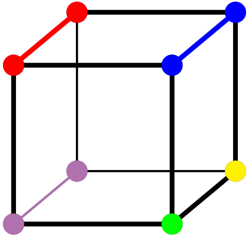

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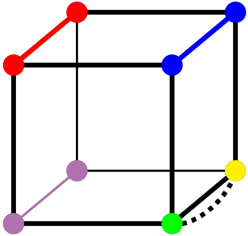

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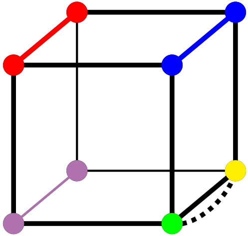

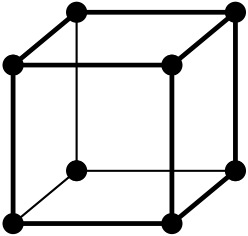

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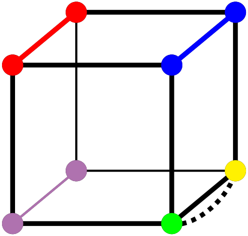

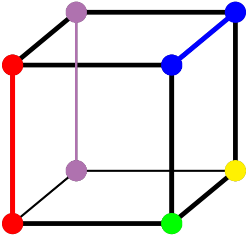

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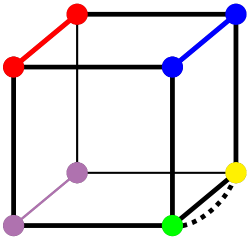

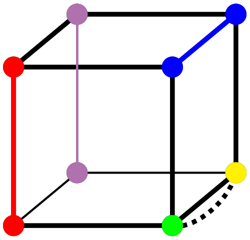

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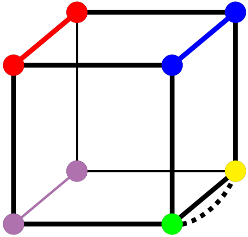

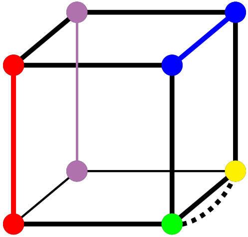

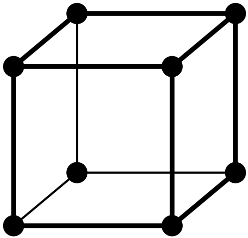

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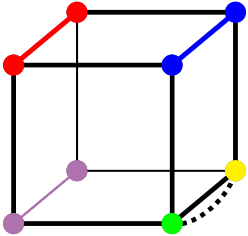

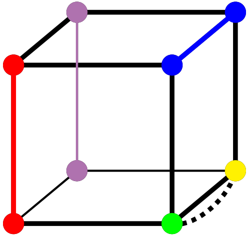

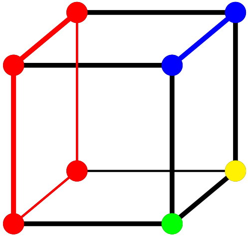

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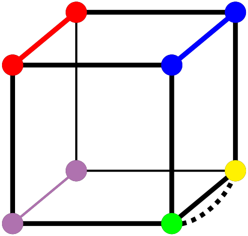

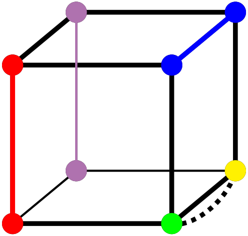

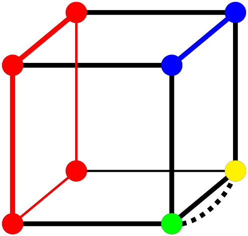

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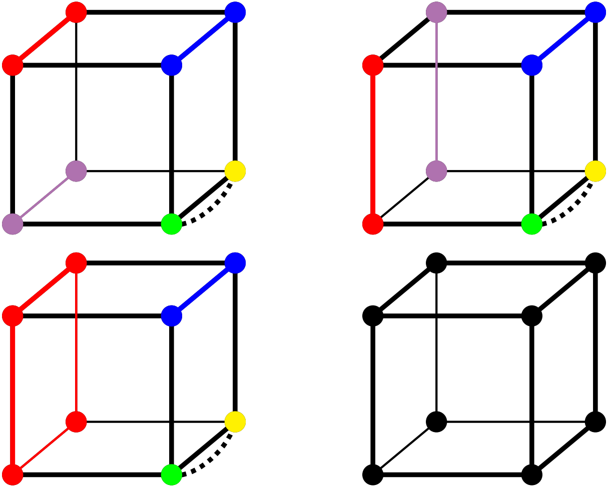

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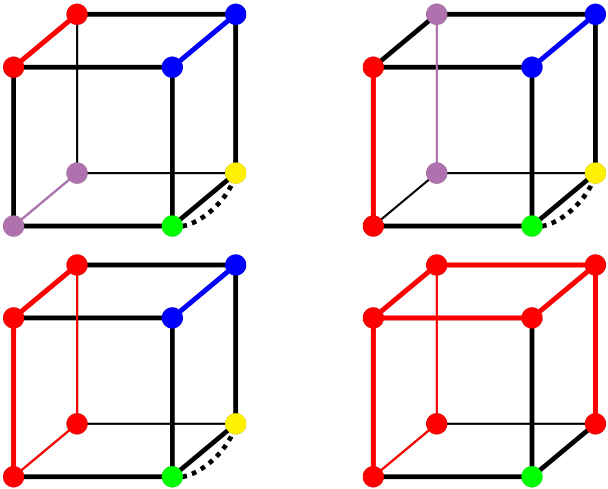

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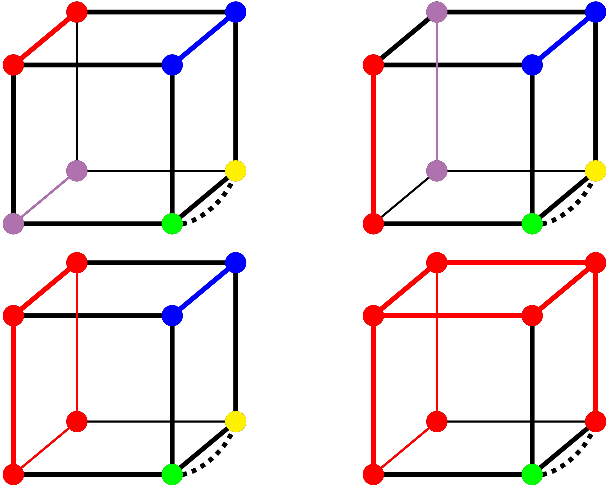

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This only makes a difference for the most pathological algebras, and for these the new definition is stronger and easier to relativize.

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Thus, supernilpotence implies nilpotence for any congruence of a finite algebra.

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The complexity of circuit satisfiability over $\mathbf{A}$ *is related to supernilpotence.*