Extending Partial Projective Planes

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Axioms for a projective plane

- Two points determine a unique line.
- Two lines intersect in a unique point.
- There exist four points with no three on a line.

A plane of order n has

- $n + 1$ points on each line.
- $n + 1$ lines through each point.
- $n^2 + n + 1$ total points.
- $n^2 + n + 1$ total lines.

Orders that are possible

Prime powers: 2,3,4,5,7,8,9,11,13,16,17,19,...

Orders that are impossible

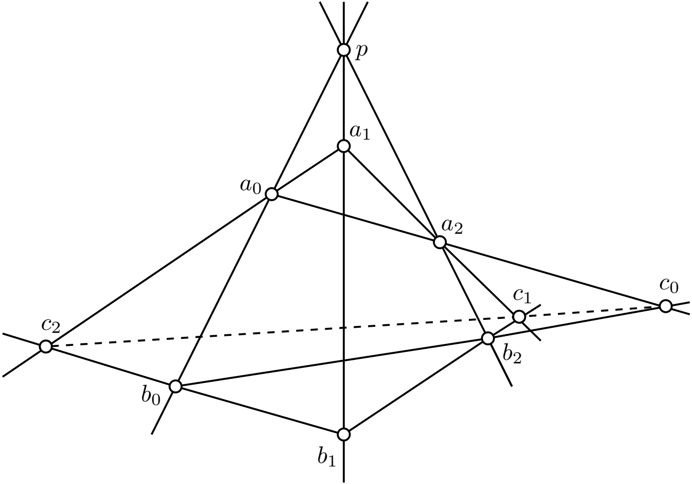
6,10,14,21,22,...

- Bruck-Ryser: If $n \equiv 1$ or $2 \bmod 4$ and there is a plane of order n, then n is a sum of two squares.
- Lam: There is no projective plane of order 10.

Orders that are unknown

12,15,18,20,...

Desargues' Law



Desarguean projective planes

A plane is **desarguean** if two triangles perspective from a point are perspective from a line.

- A plane is desarguean if and only if it can be coordinatized by a skew-field.
- So there are finite desarguean planes of order p^k for all $k \geq 1$.

Non-desarguean projective planes

- M. Hall: There are non-desarguean planes of order p^k for $k \geq 2$, $p^k \geq 9$.
- H. Neumann: Every Hall plane has a subplane of order 2.
- There are other non-desarguean planes of order p^k for $k \geq 2$.

Partial projective planes

A **partial projective plane** is a collection of points and lines, and an incidence relation, so that

- two points lie on at most one line,
- two lines intersect in at most one point.

Free extensions

M. Hall: Every finite partial plane can be extended to a projective plane (usually infinite).

Four questions

- Is there a finite projective plane of non-prime-power order (necessarily non-desarguean)?
- Is there a non-desarguean plane of prime order?
- Does every finite non-desarguean plane contain a subplane of order 2?
- Does every finite partial plane have an extension to a finite plane?

Projective planes as lattices

Projective planes correspond to simple, complemented, modular lattices of height 3.

Partial planes that are meet semilattices

A **semiplane** is a collection of lines and points, with an incidence relation, such that any two lines intersect in a unique point (AKA dual linear space).

Example

Any subset of the lines of a plane, together with the points that are intersections of those lines.

Note

Every finite partial plane can be extended to a finite semiplane.

Plan to construct a plane of order n

- Start with a semiplane that contains your desired configuration (e.g., failure of Desargues' Law or a plane of order 2)
- As long as possible add lines, with their intersections with existing lines, one at a time, keeping a semiplane structure and at most $n+1$ points-per-line.
- Intersections could be new points or old points.
- If you get n^2+n+1 lines, the semiplane is a plane.
- Otherwise, when adding a line is no longer possible, back up and try again.

Warning!

We have been doing this unsuccessfully since 1999, so be patient.

A simple turnaround criterion

Given n and a semiplane $\Pi = \langle P_0, L_0, \leq_0 \rangle$, define

$$\rho_n(\Pi) = \sum_{\ell \in L_0} r_\Pi(\ell) + n^2 + n + 1 - |P_0| - |L_0|(n+1).$$

If $\Pi = \langle P_0, L_0, \leq_0 \rangle$ can be extended to a projective plane $\Sigma = \langle P, L, \leq \rangle$ of order n, then

$$\rho_n(\Pi) = |\{p \in P : p \not\leq \ell \text{ for all } \ell \in L_0\}|.$$

Hence Π can be extended to a projective plane of order n only if $\rho_n(\Pi) \geq 0$.

Constructing nondesarguean planes

- A nondesarguean configuration has 10 points and 12 lines.
- To form a semiplane, those lines can intersect in various ways.
- The intersections could be new points or old ones.
- The result is a semiplane with 12 lines and between 20 and 37 points.
- Seffrood: There are 875 such nondesarguean semiplanes, which fall into 105 isomorphism classes.
- For some pairs (A,B) of the 105 types, if a plane contains a semiplane of type A then it contains one of type B.
- There are 15 nondesarguean semiplanes that are minimal in the sense that every nondesarguean plane must contain one of these 15 semiplanes.

Problem

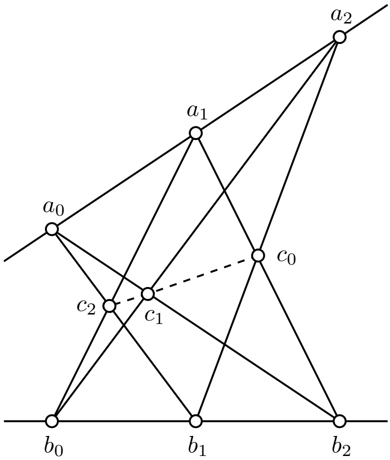
Find $f(n)$ such that every semiplane with at least $f(n)$ lines and at most $n + 1$ points-per-line can be extended to a plane of order n.

So far we have ...

- semiplanes of order 11 with 40 lines (a plane has 133 lines),
- semiplanes of order 12 with 44 lines (a plane has 157 lines),
- semiplanes of order 13 with 48 lines (a plane has 183 lines),
- semiplanes of order 15 with 56 lines (a plane has 241 lines).

These have the full number of points.

Pappus' Law



Question

Why not start with a non-pappian semiplane?

Question

What happens if you start with a plane of order 2 or 3 and try to extend it to plane of order n?

Question

What happens if you start with the configuration of a coordinatizing frame satisfying $1 + 1 + 1 + 1 = 0$? Can this partial plane be extended to a finite plane?

Compare

Freese: There are lattice equations that hold in all finite dimensional modular lattices, but not all modular lattices.

Nondesarguean planes are a different sort of animal

MAHALO!