A Primer of Subquasivariety Lattices

Kira Adaricheva, Jennifer Hyndman, J. B. Nation,
Joy N. Nishida

Hofstra University, UNBC, University of Hawai'i

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Outline

- Properties of subquasivariety lattices
- Representation theorems
- Equaclosure operators revisited: new restrictions
- A construction project
- Problems

Properties of subquasivariety lattices

A subquasivariety lattice $L_q(\mathcal{K})$ has the following properties:

- dually algebraic
- join semi-distributive - *Jónsson-Kiefer property*
- atomic
- equaclosure operators

Terminology

Let $\mathbf{S}$ be an algebraic lattice.

- A subset $X \subseteq S$ is an **algebraic subset** if it contains $1_{\mathbf{S}}$ and is closed under arbitrary meets and nonempty directed joins.
- An operator $h : S \to S$ is **continuous** if it preserves $1_{\mathbf{S}}$, arbitrary meets and nonempty directed joins.

If H is a monoid of continuous operators on $\mathbf{S}$, then $\mathrm{S_p}(\mathbf{S}, H)$ denotes the lattice of all H-closed algebraic subsets of $\mathbf{S}$, ordered by inclusion.

Representation theorem for quasivarieties

Hoehnke, AN, HNN

For a quasivariety $\mathcal{K}$, the lattice $L_q(\mathcal{K})$ is isomorphic to the lattice $\mathrm{S_p}(\mathbf{S}, H)$ where

- $\mathbf{S} = \mathrm{Con}_{\mathcal{K}}\, \mathbf{F}_{\mathcal{K}}(\omega)$
- $H = \mathcal{E}^*$ are maps derived from endomorphisms.

Main Results

- If $\mathcal{K}$ is a quasivariety, then $L_q(\mathcal{K}) \cong \mathrm{S_p}(\mathbf{S}, H)$ for an $\mathbf{S}, H$
- If $\mathbf{L}$ is finite distributive lattice, then $\mathbf{L} \cong L_q(\mathcal{K})$
- If $\mathbf{L}$ is a distributive dually algebraic lattice, then $\mathbf{L} \cong \mathrm{S_p}(\mathbf{S}, H)$
- $(\omega + 1)^d \ncong L_q(\mathcal{K})$
- If $\mathbf{L} \cong \mathrm{S_p}(\mathbf{S}, H)$, then $1 + \mathbf{L} \cong L_q(\mathcal{K})$

Classical properties of equaclosure operators (DAG)

Let **L** be a dually algebraic lattice.

(I1) $x \leq \gamma(x)$.

(I2) $x \leq y$ implies $\gamma(x) \leq \gamma(y)$.

(I3) $\gamma^2(x) = \gamma(x)$.

(I4) $\gamma(0) = 0$.

(I5) $\gamma(x) = u$ for all $x \in X$ implies $\gamma(\bigwedge X) = u$.

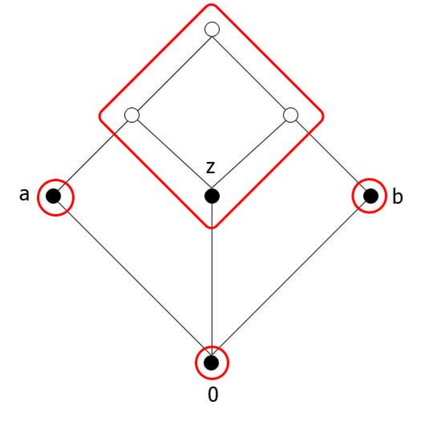
(I6) $\gamma(x) \wedge (y \vee z) = (\gamma(x) \wedge y) \vee (\gamma(x) \wedge z)$.

(I7) $\gamma(L)$ is the complete meet subsemilattice of **L** generated by $\gamma(L) \cap K$, the semilattice of dually compact elements.

(I8) *There is a dually compact element $w \in L$ such that $\gamma(w) = w$ and the interval $[0, w]$ is isomorphic to $\mathrm{S_p}(\mathbf{S})$ for some algebraic lattice **S**.

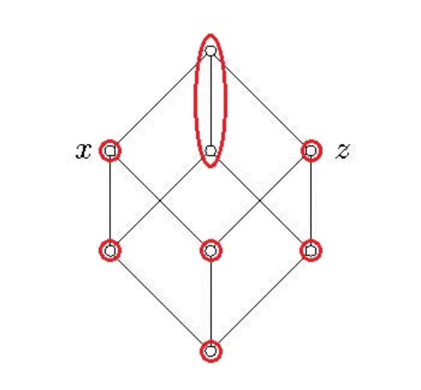
Define τ

- (I5) defines $\tau(x)$ abstractly
- $\tau(a \vee b) \leq \tau(a) \vee \tau(b)$



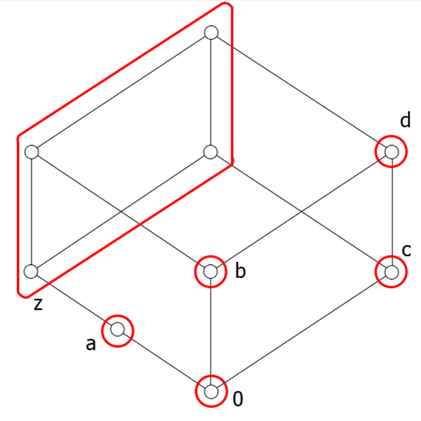
Condition (K9)

$$\gamma[x \wedge \tau(x \vee z)] \geq x \wedge \tau(z)$$



Condition (K10)

$$\tau b \leq \tau d \ \&\ \gamma c \leq \gamma d \leq \gamma(a \vee c) \ \&\ c \wedge \gamma(b) \leq \gamma a \rightarrow \gamma b \leq \gamma a$$



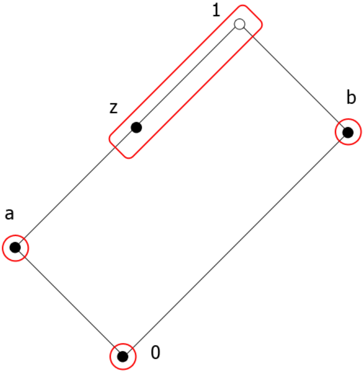
Construction project

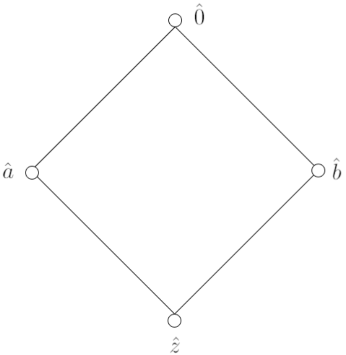
Given a pair $(\mathbf{L}, \gamma)$ with $\mathbf{L}$ a finite lower bounded lattice and γ satisfying the known properties of natural equaclosure operators, can we represent $\mathbf{L}$ as

- $\mathrm{S_p}(\mathbf{S}, H)$
- $L_q(\mathcal{K})$ for a quasivariety of structures

with γ corresponding to the natural equaclosure operator?

Six-step program - Step 1: $\mathbf{L} = \mathrm{Sub}(\mathbf{S}, \wedge, \hat{0}, h)$





$$\hat{z} \to \hat{a}$$
$$\hat{a} \,\&\, \hat{b} \to \hat{z}$$

	h
$\hat{0}$	$\hat{0}$
$\hat{a}$	$\hat{a}$
$\hat{b}$	$\hat{0}$
$\hat{z}$	$\hat{a}$

Steps 2–5 (routine): $\mathbf{L} = L_q(\mathcal{K}_0)$

Convert $\mathrm{Sub}(\mathbf{S}, \wedge, \hat{0}, h)$ to $L_q(\mathcal{K}_0)$ in a language without equality.

$\mathcal{K}_0$ has the operations e, μ and predicates O, A, B with laws

$$P(e), \quad P(\mu e) \qquad \text{for } P = O, A, B$$

$$P(\mu^2 x) \leftrightarrow P(\mu x) \qquad \text{for } P = O, A, B$$

$$O(x) \to A(x)$$

$$O(x) \to B(x) \leftrightarrow B(\mu x) \leftrightarrow O(\mu x)$$

$$A(\mu x)$$

Step 6: $\mathbf{L} = L_q(\mathcal{K}_1)$

Convert $\mathcal{K}_0$ to a quasivariety $\mathcal{K}_1$ with equality (if possible) by interpreting $O(x)$ as $x \approx e$.

$\mathcal{K}_1$ has the operations e, μ and a predicate A with laws

$$A(e) \qquad \mu e \approx e$$

$$\mu^2 x \approx \mu x \qquad A(\mu x)$$

$$A(x) \,\&\, \mu x \approx e \to x \approx e$$

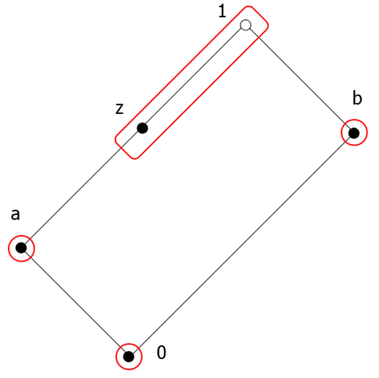
$$A(x) \to \mu x \approx x$$

Step 6 simplified: $\mathbf{L} = L_q(\mathcal{K}_2)$

With the interpretation $A(x) \mapsto \mu x \approx x$ we obtain an equivalent quasivariety $\mathcal{K}_2$ with operations e, μ and the laws

$$\mu e \approx e$$
$$\mu^2 x \approx \mu x$$

$L_q(\mathcal{K}_2)$



0: $x \approx e$ a: $\mu x \approx x$ b: $\mu x \approx e$ z: $\mu x \approx e \to x \approx e$
1: $\mu e \approx e$, $\mu^2 x \approx \mu x$

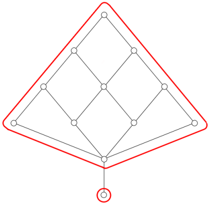
$J \subseteq T$ Theorem

Let $(\mathbf{L}, \gamma)$ be a finite, lower bounded lattice with a weak equaclosure operator. If $(\mathbf{L}, \gamma)$ satisfies $J \subseteq T$, then $(\mathbf{L}, \gamma)$ has a representation as $\mathrm{S_p}(\mathbf{S}, H)$ if and only if there exists a set of operators H^* on $\gamma(\mathbf{L})$ satisfying the conditions below. If such a set of operators exists, then $\mathbf{L} \cong \mathrm{Sub}(\gamma(\mathbf{L}), 0, \vee, H^*)$.

1. $h^*[a] \leq [a]$.
2. $\tau h^*[a] \leq \tau[a]$.
3. $h^*[0] = [0]$.
4. $[c] \leq [d]$ implies $h^*[c] \leq h^*[d]$.
5. $h^*(\bigvee_i [r_i]) = \bigvee_i h^*[r_i]$.
6. $\tau[a] \leq \tau[b]$ implies there exists $k \in \mathbb{DMO}$ such that $k^*[b] = [a]$.

Problems

- Find more restrictions on pairs $(\mathbf{L}, \gamma)$ to be representable.
- Finish the construction project.
- Decide the test cases: Can you represent $\text{Fin}(X) + \mathbf{1}$ as $\text{S}_\text{p}(\mathbf{S}, H)$, where X is an infinite set?
- Can you represent the leaf $\mathbf{1} + \text{Co}(\mathbf{4})$ as $L_q(\mathcal{K})$ in a language with equality?



Thank you

MAHALO!