# The role of twisted wreath products in the finite congruence lattice problem


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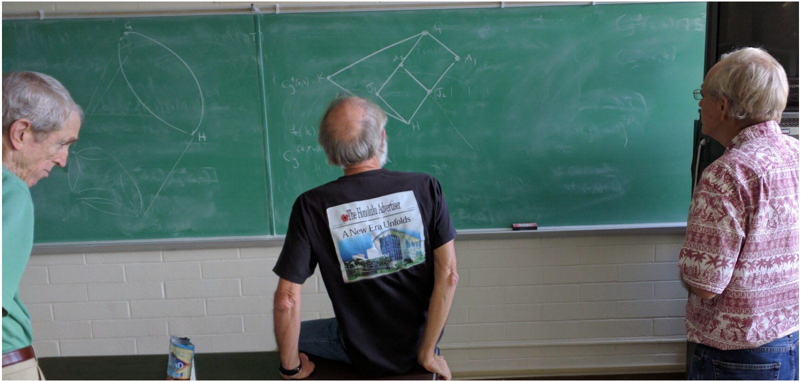
# What are they looking at?



# The Grätzer–Schmidt Theorem

**Theorem.** (G. Grätzer* and E. T. Schmidt, 1963) For every algebraic lattice $L$ there exists an algebra $A$ with $\mathrm{Con}(A) \cong L$.

* Grätzer was Bill's thesis advisor at Penn State.

All known proofs of the Grätzer–Schmidt Theorem construct an infinite algebra for (most) finite lattices: Grätzer and Schmidt, 1963; Lampe, 1973; Pudlák, 1976; Tůma, 1989.

So the problem arises:

**Open Problem.** Is every finite lattice isomorphic to the congruence lattice of a finite algebra?

# Cayley's Theorem

Let $G$ be group. One constructs a multi-unary algebra $(G; G)$ where each operation $g \in G$ is a permutation $x \mapsto xg$ $(x \in G)$.
This is the construction in Cayley's Theorem:
Every (abstract) group is isomorphic to a permutation group.

What are the congruences of the multi-unary algebra $(G; G)$ ?

Let $\equiv$ be a congruence, $e \in G$ the identity element, and $H$ the equivalence class of $e$.
$H$ is a subgroup: If $a, b \in H$, then
$(e \equiv a \,\&\, b \equiv e) \Rightarrow a^{-1}b = e(a^{-1}b) \equiv a(a^{-1}b) = b \equiv e$.
The congruence classes of $\equiv$ are the right cosets of $H$:
$x \equiv y \Leftrightarrow xy^{-1} \equiv yy^{-1} = e \Leftrightarrow xy^{-1} \in H \Leftrightarrow x \in Hy \Leftrightarrow Hx = Hy$.
Hence $\mathrm{Con}(G; G) \cong \mathrm{Sub}(G)$, the subgroup lattice of $G$.
Furthermore $\mathrm{Con}(G/\equiv_H; G) \cong \mathrm{Int}(H, G) = \{X \mid H \leq X \leq G\}$, the **interval** between $H$ and $G$ in $\mathrm{Sub}(G)$.

# Reduction to finite groups

Assume that a finite lattice can be represented as $\mathrm{Con}(A; F)$ for some finite algebra. We will always take a smallest representation, i.e., when $|A|$ is minimal.

For lattices with some specific properties (simplicity, etc.) it can be deduced that the smallest algebra (if there is any) with such a congruence lattice is actually a multi-unary algebra equivalent to a transitive permutation group, hence the congruence lattice is isomorphic to $\mathrm{Int}(H, G)$ for a finite group $G$ and a subgroup $H$.

Since every finite lattice can be embedded as an interval into a finite lattice with the required properties, we have the first reduction theorem:

# $P^5$

**Theorem.** ($P^3$ and P. Pudlák, 1980) The following statements are equivalent:
(1) Every finite lattice is isomorphic to the congruence lattice of a finite algebra.
(2) For every finite lattice $L$ there exist a finite group $G$ and a subgroup $H$ such that the **interval sublattice** $\mathrm{Int}(H, G) = \{X \mid H \leq X \leq G\}$ in the subgroup lattice of $G$ is isomorphic to $L$.

Note that in case when these statements are false, we do not claim that every congruence lattice of a finite algebra can be represented as an interval in the subgroup lattice of a finite group.

# Further reduction

**Main Theorem.** (mainly due to Ferdinand Börner, 1999; similar ideas are contained in papers of R. Baddeley, A. Lucchini, J. Shareshian, M. Aschbacher)

The following statements are equivalent:
(2) Every finite lattice can be represented as an interval $\mathrm{Int}(H, G)$, where $G$ is a finite group and $H$ is a subgroup of $G$.

(3)
*either* (3a) every finite lattice can be represented as an interval $\mathrm{Int}(H, G)$ where $G$ is a finite **almost simple group**, and $H$ is a **core-free subgroup** of $G$.
*or* (3b) every finite lattice can be represented as an interval $\mathrm{Int}(H, G)$ where $G$ is a **twisted wreath product** with ingredients $H$, $T$, $H_1 < H$, $\varphi : H_1 \to \mathrm{Aut}(T)$,
where $H$ is a finite group, $T$ is a finite non-abelian simple group, $H_1$ is a core-free subgroup of $H$, and $\varphi(T_1)$ contains all inner automorphisms of $T$.

# Sources

R. Baddeley, A. Lucchini, On representing finite lattices as intervals in subgroup lattices of finite groups, *Journal of Algebra* 196 (1997), 1-100.

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J. Shareshian, Topology of order complexes of intervals in subgroup lattices, *Journal of Algebra* 268 (2003), 677–686.

M. Aschbacher, On intervals in subgroup lattices of finite groups, *Journal of the American Mathematical Society* 21 (2008), 809–830.

# Almost simple groups

(3a) every finite lattice can be represented as an interval $\mathrm{Int}(H, G)$ where $G$ is a finite **almost simple group**, and $H$ is a **core-free subgroup** of $G$.

A group $G$ is **almost simple**, if it has a simple normal subgroup $S$ with trivial centralizer $\mathbf{C}_G(S) = 1$.
Conjugation by elements of $G$ yield automorphisms of $S$, so we get an embedding $G \to \mathrm{Aut}(S)$. The image contains all inner automorphisms of $S$, i.e., (up to isomorphism)

$$S \cong \mathrm{Inn}(S) \leq G \leq \mathrm{Aut}(S).$$

Reduction to almost simple groups is a standard method in group theory. After that one tries to solve the problem for the various types of finite simple groups, relying on the **Classification of Finite Simple Groups (CFSG)**.

# The Schreier Hypothesis

In our case the reduction itself uses the CFSG through one of its best known consequences, the **Schreier Hypothesis** stating that the **outer automorphism group**

$$\mathrm{Out}(S) = \mathrm{Aut}(S)/\mathrm{Inn}(S)$$

of any finite simple group $S$ is a solvable group.

The **core** of a subgroup $H < G$ is the largest normal subgroup of $G$ contained in $H$, namely

$$\bigcap_{g \in G} g^{-1} H g.$$

$H$ is **core-free**, if $\bigcap_{g \in G} g^{-1} H g = 1$.

# The standard wreath product

Ingredients:
$D$ — finite group *("domain")*
$T$ — arbitrary group (often non-abelian simple) *("target")*

$F$ = all functions $f : D \to T$ with pointwise multiplication,
$F \cong T^{|D|}$

$D$ acts on $F$ by $\boxed{f^d(x) = f(xd^{-1})}$
Indeed, $f^{d_1 d_2}(x) = f(x(d_1 d_2)^{-1}) = f((xd_2^{-1})d_1^{-1}) = f^{d_1}(xd_2^{-1}) = (f^{d_1})^{d_2}(x)$.

The semidirect product $F \rtimes D$ is the **standard wreath product** $T \wr D$.

# Subdirect products of simple groups

A subgroup $H \le G_1 \times G_2 \times \cdots \times G_n$ is a **subdirect product** if all projections $H \to G_i$ $(i = 1, 2 \ldots, n)$ are surjective.

**Lemma.** Let $T$ be a non-abelian simple group, and let $H \le T^n$ be a subdirect product. Then $H \cong T^m$ for some $1 \le m \le n$. Moreover, there is a map $\nu : \{1, \ldots, n\} \twoheadrightarrow \{1, \ldots, m\}$ and automorphisms $\varphi_i \in \mathrm{Aut}(T)$ such that $f : \{1, \ldots, n\} \to T$ belongs to $H$ iff $\boxed{f(i) = \varphi_i(t_{\nu(i)})}$ with $t_1, \ldots, t_m \in T$.

An example: $H = \{(t_1, \varphi_2(t_1), t_2, \varphi_4(t_1), t_2) \mid t_1, t_2 \in T\} < T^5$.

So there is a partition $I_1 \cup \cdots \cup I_m = \{1, \ldots, n\}$ and for each class a parameter $t_j \in T$ such that the coordinates for elements of $H$ corresponding to the indices in the class $I_j$ are determined by $t_j$, namely, they are images of $t_j$ under a suitable automorphism of $T$, that is fixed independently for each coordinate.

# $D$-invariant subgroups of $F$ (1)

**Question.** Assume that $T$ is a non-abelian simple group. What are the $D$-invariant subdirect products in $F \cong T^{|D|}$ ?

Let $H$ be a $D$-invariant subdirect product determined by $\nu : D \twoheadrightarrow \{1, \ldots, m\}$ and $\varphi_x \in \mathrm{Aut}(T)$ $(x \in D)$.
Then $f \in H$ iff $\forall x \in D : f(x) = \varphi_x(t_{\nu(x)})$.

Let us fix $b \in D$, then by the invariance of $H$ we have $f^{b^{-1}} \in H$, i.e., $\forall x \in D : f^{b^{-1}}(x) = \varphi_x(u_{\nu(x)})$ with $u_1, \ldots, u_m \in T$.
The partition corresponding to $\nu$ is invariant under right translations by elements of $D$, hence it is a partition into right cosets of some subgroup $D_1$, so $D = D_1x_1 \cup D_1x_2 \cup \cdots \cup D_1x_m$ with $\nu(d) = i$ iff $d \in D_1x_i$.
W.l.o.g. $x_1 = 1$ (so $\nu(1) = 1$) and $\varphi_{x_i} = \mathrm{id}$ for each $i = 1, \ldots, m$.

# $D$-invariant subgroups of $F$ (2)

Let us assume that $b \in D_1 x_i$ and $a \in D_1$, so $ab \in D_1 x_i$ as well.
Then $f(ab) = f^{b^{-1}}(a) = \varphi_a(u_{\nu(a)}) = \varphi_a(u_1)$. For $a = 1$ this yields $u_1 = f(b)$, hence $\boxed{\forall a \in D_1, \forall b \in D : f(ab) = \varphi_a(f(b))}$.
For $b = 1$ it yields $\forall a \in D_1 : f(a) = \varphi_a(f(1))$.
If $a, b \in D_1$, then $\varphi_{ab}(f(1)) = f(ab) = \varphi_a(f(b)) = \varphi_a(\varphi_b(f(1)))$.
Since $H$ is a subdirect product, $f(1)$ can be any element of $T$, so $\varphi : D_1 \to \mathrm{Aut}(T)$ is a homomorphism.
Furthermore, $\varphi_{ax_i}(t_i) = f(ax_i) = \varphi_a(f(x_i)) = \varphi_a(t_i)$, so $\forall a \in D_1, \forall i \in \{1, \ldots, m\} : \varphi_{ax_i} = \varphi_a$.

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For $b = 1$ it yields $\forall a \in D_1 : f(a) = \varphi_a(f(1))$.

If $a, b \in D_1$, then $\varphi_{ab}(f(1)) = f(ab) = \varphi_a(f(b)) = \varphi_a(\varphi_b(f(1)))$.

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Conversely, it is easy to verify that if $D_1 \leq D$, $D = D_1 x_1 \cup \cdots \cup D_1 x_m$, and $\varphi : D_1 \to \mathrm{Aut}(T)$ is a homomorphism, then

$H = \{f : D \to T \mid f(ax_i) = \varphi_a(t_i), a \in D_1, t_i \in T (i = 1, \ldots, m)\}$

is a $D$-invariant subdirect product in $F \cong T^{|D|}$.

# Twisted wreath products (d'après Suzuki)

Introduced by B. H. Neumann in 1963

Ingredients:

$D$ — (finite) group

$T$ — arbitrary group

$D_1 \leq D$ — a subgroup of $D$

$\varphi : D_1 \to \mathrm{Aut}(T)$ — a homomorphism into the automorphism group of $T$

$F_1 = \{f : D \to T \mid \forall x \in D_1, y \in D : \boxed{f(xy) = \varphi_x(f(y))}\}$

$F_1 \cong T^{|D:D_1|}$: Choose representatives of right cosets $D = D_1 y_1 \cup \cdots \cup D_1 y_m$ and for $d \in D_1$ let $f(dy_i) = \varphi_d(t_i)$. If $x \in D_1$, then $f(x(dy_i)) = \varphi_{xd}(t_i) = \varphi_x(\varphi_d(t_i)) = \varphi_x(f(dy_i))$, so $f \in F_1$.

$F_1$ is invariant under the action of $D$, since for any $b \in D$, if $f \in F_1$, $x \in D_1$, $y \in D$, then $f^b(xy) = f(xyb^{-1}) = \varphi_x(f(yb^{-1})) = \varphi_x(f^b(y))$, so $f^b \in F_1$.

The **twisted wreath product** is the semidirect product $F_1 \rtimes D$.

# A problem for group theorists

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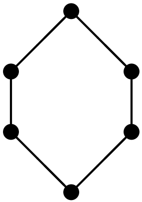
If $F \subset E$ is a finite, separable field extension and $F \subset E \subset E^*$ is the smallest Galois extension then the lattice of intermediate fields $\{X \mid F \subseteq X \subseteq E\}$ is dually isomorphic to $\mathrm{Int}(\mathrm{Gal}(E^*|E), \mathrm{Gal}(E^*|F))$.

# Operator algebras

**Open Problem.** Is every finite lattice isomorphic to the lattice of intermediate subfactors of a von Neumann algebra?

**Theorem.** (Y. Watatani, 1996) If a lattice can be represented as an interval in a subgroup lattice of a finite group, then it also occurs as a lattice of intermediate subfactors of a von Neumann algebra.

For example, is the hexagon lattice isomorphic to the lattice of intermediate subfactors of a von Neumann algebra?



# Subgroups of twisted wreath products (1)

Let $G = F_1 \rtimes D$ be a twisted wreath product with ingredients satisfying the following assumptions:
$D$ — a finite group
$T$ — a finite non-abelian simple group
$D_1$ — a subgroup of $D$
$\varphi : D_1 \to \mathrm{Aut}(T)$ — a homomorphism such that $\varphi(D_1) \geq \mathrm{Inn}(T)$
Note that $\mathrm{Inn}(T) \cong T$ is the unique minimal normal subgroup of $\mathrm{Aut}(T)$.

We are going to determine $\mathrm{Int}(D, G)$. If $D \leq X \leq G$, then $X = (F_1 \cap X)D$ and $F_1 \cap X$ is a $D$-invariant subgroup. Conversely, if $H \leq F_1$ is $D$-invariant, then $D \leq HD \leq G$. Thus $\mathrm{Int}(D, G) \cong \mathrm{Sub}_D(F_1)$, the lattice of $D$-invariant subgroups of $F_1$.

# Subgroups of twisted wreath products (2)

Let $H \leq F_1$ be a $D$-invariant subgroup. Denote by $U \leq T$ the projection of $H$ to the first component, and let $u \in U$ (so $u = f(1)$ for some $f \in H$), $a \in D_1$. Then $\varphi_a(u) = \varphi_a(f(1)) = f(a) = f^{a^{-1}}(1) \in U$, hence $U$ is a $\varphi(D_1)$-invariant subgroup of the simple group $T$. By assumption, $\varphi(D_1) \geq \mathrm{Inn}(T)$, so either $U = 1$ or $U = T$. Since $D$ acts transitively on the components, either $H = 1$ or $H$ is a subdirect product.

In the latter case $H$ is determined by a subgroup $D_2 \leq G$ and a homomorphism $\psi : D_2 \to \mathrm{Aut}(T)$. In order $H$ to be contained in $F_1$ we need $D_2 \geq D_1$ and $\psi\big|_{D_1} = \varphi$.

We have obtained:

**Proposition.** $\mathrm{Int}(D, G) \cong \mathrm{Sub}_D(F_1)$ is dually isomorphic to the lattice of all extensions of $\varphi$ from $D_1$ to subgroups of $D$ together with an additional top element (corresponding to $H = 1$).

Aschbacher calls it a **signalizer lattice**.

# An example (1)

M. Aschbacher (2008)
Let $T = A_5$, $D = A_6 \times A_6$,
$D_1 = \mathrm{diag}(A_5) = \{(x, x) \mid x \in A_5\} \leq D$, $\varphi : D_1 \to \mathrm{Inn}(T)$ an isomorphism, and $G$ the twisted wreath product constructed from these ingredients. We know that $\mathrm{Int}(D, G) \cong \mathrm{Sub}_D(T^{|D:D_1|})$ is dually isomorphic to the signalizer lattice for these data.

What are the subgroups of $D$ containing $D_1$?

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What are the subgroups of $D$ containing $D_1$?
$\mathrm{diag}(A_5)$, $A_5 \times A_5$, $A_5 \times A_6$, $A_6 \times A_5$, $\mathrm{diag}(A_6)$, $A_6 \times A_6$.

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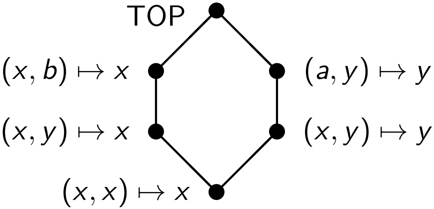
$\mathrm{diag}(A_5)$, $A_5 \times A_5$, $A_5 \times A_6$, $A_6 \times A_5$, $\mathrm{diag}(A_6)$, $A_6 \times A_6$.

How can we extend $\varphi$ to these subgroups?
Two ways to $A_5 \times A_5$: using either the first or the second projection,
one way to $A_5 \times A_6$ using the first projection,
one way to $A_6 \times A_5$ using the second projection,
and it cannot be extended to the last two subgroups.

# An example (2)

Hence the dual of the interval $\mathrm{Int}(D, G)$ is the hexagon.
$x, y \in A_5,\ a, b \in A_6$



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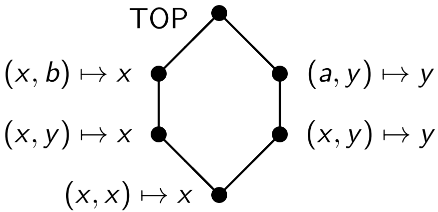
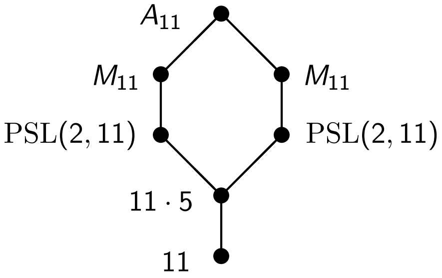



**Exercise.** Let $T = A_5$, $D = S_5 \times A_5$, $D_1 = \mathrm{diag}(A_5) \leq D$, $\varphi : D_1 \to \mathrm{Inn}(T)$ an isomorphism, and $G$ the twisted wreath product constructed from these ingredients. Determine the interval $\mathrm{Int}(D, G)$.

# Referee's remark

An example for the hexagon as an interval in the subgroup lattice of a finite group can be found in a paper published much earlier than Aschbacher's. Namely,




Here $A_{11}$ is the alternating group of degree 11,
$M_{11}$ is the Mathieu group of degree 11 (the smallest sporadic simple group),
$\mathrm{PSL}(2, 11)$ is a projective special linear group (Galois observed that it has an action of degree 11)
11 denotes a cyclic group of order 11, and $11 \cdot 5$ its normalizer.

# The reference

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P. P. Pálfy, On Feit's examples of intervals in subgroup lattices, *Journal of Algebra* 116 (1988), 471–479.

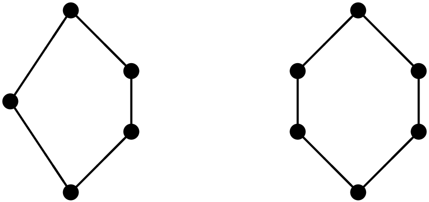
# Quasiprimitive permutation groups

If $N \lhd G$, $N \leq H$, then $\mathrm{Int}(H, G) \cong \mathrm{Int}(H/N, G/N)$, so w.l.o.g. we may assume that $H$ is core-free, i.e., $\bigcap_{g \in G} g^{-1}Hg = 1$. We may think of $G$ as a permutation group with $H$ being one of the point-stabilizers.

The permutation group $G$ with point stabilizer $H$ is **quasiprimitive** if every non-trivial normal subgroup $N$ of $G$ is transitive, that is $NH = G$.

# Strongly non-modular lattices

We call a lattice $L$ **strongly non-modular** if for every $y \in L \setminus \{0_L, 1_L\}$ there exist $x < z$ such that $x \vee (y \wedge z) \neq (x \vee y) \wedge z$.



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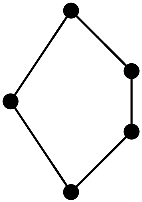
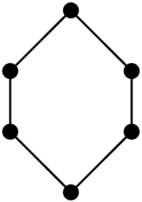


non-modular



strongly non-modular

# A lemma

**Lemma.** If $\mathrm{Int}(H, G)$ is strongly non-modular, then for every normal subgroup $N \lhd G$ either $N \leq H$ or $NH = G$ holds.
So if $H$ is core-free, then the permutation group $G$ with stabilizer $H$ is quasiprimitive.
*Proof.* Assume the contrary, that is, for some $1 \neq N \lhd G$ we have $X = NH < G$. $H$ is core-free, hence $X = NH > H$.

Take the subgroups $H < Y < Z < G$ satisfying $X \vee Y = X \vee Z$, $X \wedge Y = X \wedge Z$.
Now $X \vee Y = \langle X, Y \rangle \supseteq XY = NHY = NY$.
Here $NY$ is a subgroup, hence $XY = \langle X, Y \rangle$.
Furthermore, $X \vee Y = X \vee Z \supseteq XZ \supseteq XY = X \vee Y$, so $XZ = XY$.
Let $z \in Z \leq XY$, then $z = xy$ with $x \in X$, $y \in Y$. Now $x = zy^{-1} \in X \cap Z = X \cap Y$, so $z = xy \in Y$, contradicting $Y < Z$.

# The minimal normal subgroup of $G$

Let $M$ be a minimal normal subgroup of $G$. Then $\mathrm{Int}(H, G) \cong \mathrm{Int}_H(H \cap M, M)$.

If $M$ is an elementary abelian $p$-group, then $\mathrm{Int}_H(H \cap M, M)$ is a modular lattice.
If $M$ is a simple group, then $G$ is almost simple.
If $M$ is the direct product of $k \geq 2$ isomorphic copies of a non-abelian simple group $T$, then any maximal $H$-invariant subgroup of $M$ is either a direct product of $k$ copies of a subgroup of $T$ or it is a subdirect product in $T^k$. (Cf. the proof of the O'Nan–Scott Theorem.)

Certain lattice theoretic properties of $\mathrm{Int}_H(H \cap M, M)$ imply that every subgroup in this interval is a subdirect product.

# Use Twisted Wreath Products

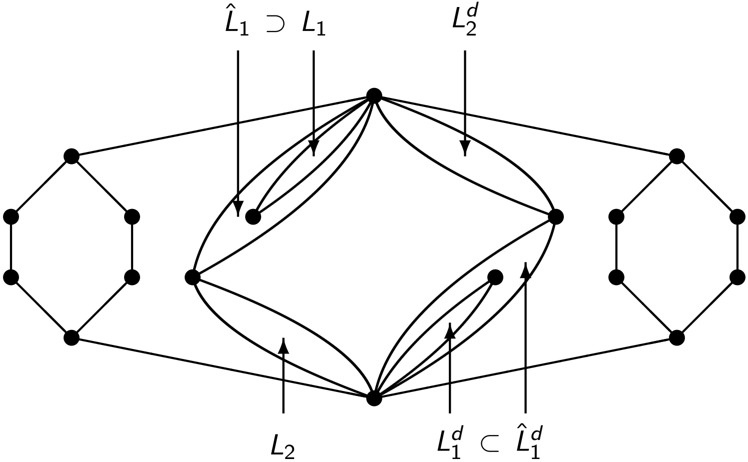
Aschbacher's paper gives a similar dichotomy for representing certain types of lattices (what he calls CD-lattices). He does not consider lattice embeddings to extend his result to arbitrary finite lattices. Curiously, he does not use the name *twisted wreath product*.

Börner mentions twisted wreath products only in a remark, referring to Baddeley's paper, but says "In our paper we make no use of this notion, because we feel that this would require an introduction with almost the same effort as in this section."

# The key lattice (1)

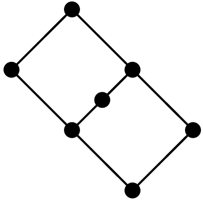
Let $L_1$ and $L_2$ be finite lattices, $\hat{L}_1$ the extension of $L_1$ that is generated by its coatoms and contains $L_1$ as a filter.
If the following lattice can be represented as an interval in the subgroup lattice of a finite group, then either $L_1$ can be represented with an almost simple group as in (3a) or $L_2$ can be represented with a twisted wreath product as in (3b).

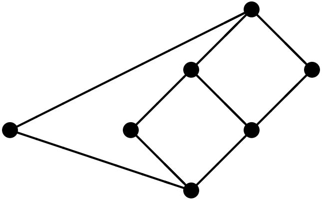
# The key lattice (2)



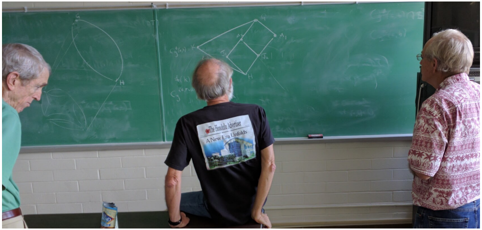
# Small lattices

W. DeMeo (PhD Thesis, University of Hawaii, 2012) found representations of lattices with $\leq 7$ elements, with two exceptions



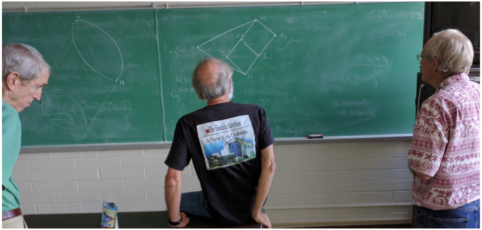


# Talk ends




**Bill, JB, Ralph,**

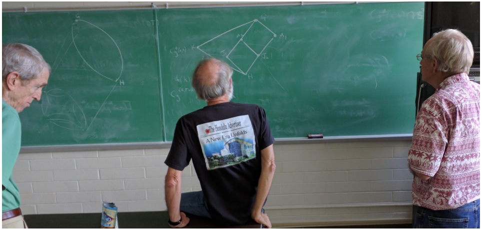
# Talk ends




**Bill,** **JB,** **Ralph,**

**Congratulations for your wonderful achievements in Universal Algebra and Lattice Theory.**

# Talk ends



**Bill,** **JB,** **Ralph,**

**Congratulations for your wonderful achievements in Universal Algebra and Lattice Theory.**

Thank you for your attention.