Ω-algebras

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Abstract

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Starting with Ω-sets where Ω is a complete lattice, we introduce the notion of an Ω-algebra. This is a classical algebra equipped with an Ω-valued equality replacing the ordinary one. In these new structures identities hold as appropriate lattice-theoretic formulas. Our investigation is related to weak congruences of the basic algebra to which a generalized equality is associated. Namely every Ω-algebra uniquely determines a closure system in the lattice of weak congruences of the basic algebra. By this correspondence we formulate a representation theorem for Ω-algebras.

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Ω-sets have been further applied to non-classical predicate logics, and also partially in theoretical foundations of fuzzy set theory.

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A generalized equality was further used in particular by Demirci (2003), Bělohlávek and Vychodil (2006) and others.

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Identities for lattice-valued structures with a fuzzy equality were introduced by Bělohlávek (2006) with graded satisfiability.
In our approach, an identity holds if the corresponding lattice-theoretic formula is fulfilled. An identity may hold on a lattice-valued algebra, while the underlying classical algebra need not satisfy the same identity.

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The collection $\mathrm{Con_w}(\mathcal{A})$ of all weak congruences on an algebra $\mathcal{A}$ is an algebraic lattice under inclusion.

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For $p \in L$, a **cut set** or a ***p*-cut** of an Ω-valued function $\mu : A \to \Omega$ is a subset μ_p of A which is the inverse image of the principal filter $\uparrow p$ in Ω: $\mu_p = \mu^{-1}(\uparrow p) = \{x \in X \mid \mu(x) \geqslant p\}$.

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An **Ω-valued** (binary) **relation R on A** is an Ω-valued function on A^2, i.e., it is a mapping $R : A^2 \to \Omega$. As above, for $p \in \Omega$, a **cut** R_p of R is the binary relation on A, which is the inverse image of $\uparrow p$: $R_p = R^{-1}(\uparrow p)$.

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Strictness can be understood as a *weak reflexivity* of R. Therefore, a symmetric and transitive Ω-valued relation on A is a **weak Ω-valued equivalence on A**.

If $\mu : A \to \Omega$ is an Ω-valued function on A, then the map $R : A^2 \to \Omega$ on A is an **Ω-valued relation on μ** if for all $x, y \in A$

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If $R : A^2 \to \Omega$ is a weak Ω-valued equivalence on A, then it is an Ω-valued equivalence on $\mu : A \to \Omega$, such that $\mu(x) = R(x, x)$. The Ω-valued function μ is said to be **determined** *by R.*

A weak Ω-valued equivalence R on A is a weak **Ω-valued equality**, if it satisfies the **separation property**:

$$\text{If} \;\; R(x,x) \neq 0, \;\; \text{then} \;\; R(x,y) = R(x,x) \;\; \text{implies} \;\; x = y.$$

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Analogously, an Ω-valued equivalence on $\mu : A \to \Omega$ satisfying the separation property is an **Ω-valued equality** on μ.

If $A = (A, F)$ is an algebra and $\mu : A \to \Omega$ an Ω-valued function on A, then μ is **compatible** with the operations in F, if for every n-ary operation $f \in F$, for all $a_1, \dots, a_n \in A$, and for every constant (nullary operation) $c \in F$

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Analogously, an Ω-valued relation $R : A^2 \to \Omega$ on A is **compatible** with the operations in F if for every n-ary operation $f \in F$, for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$, and for every constant $c \in F$

$$\bigwedge_{i=1}^{n} R(a_i, b_i) \leqslant R(f(a_1, \dots, a_n), f(b_1, \dots, b_n)), \text{ and } R(c, c) = 1.$$

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Lemma

Let $\mathcal{A} = (A, F)$ be an algebra.
An Ω-valued function $\mu : A \to \Omega$ on A is compatible with all the operations in F, if and only if for every $p \in \Omega$, μ_p is a subalgebra of $\mathcal{A}$.
Similarly, an Ω-valued relation $R : A^2 \to \Omega$ on A is compatible with all the operations in F, if and only if for every $p \in \Omega$, R_p is compatible with all the operations in F.

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As defined above, the Ω-valued function $\mu : A \to \Omega$ on A, given by $\mu(x) = E(x,x)$, is determined by E, which is a weak Ω-valued equivalence on A. But E is also an Ω-valued equality on μ.

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Lemma

Every cut $E_p = E^{-1}(\uparrow p)$, $p \in \Omega$, of the Ω-valued equality E in an Ω-set (A, E) is an equivalence relation on the corresponding cut μ_p of μ.

A pair $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is an **Ω-algebra** if $\mathcal{A} = (A, F)$ is an algebra, (A, E) is an Ω-set and E is compatible with the operations in F. $\mathcal{A}$ is the **underlying, basic algebra** of $\overline{\mathcal{A}}$.

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(i) The Ω-valued function μ determined by E is compatible with the fundamental operations on $\mathcal{A}$.
(ii) For every $p \in \Omega$, the cut μ_p of μ is a subalgebra of $\mathcal{A}$, and
(iii) Every cut of E is a weak congruence on $\mathcal{A}$, namely for $p \in E$, E_p is a congruence on μ_p.

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Let $(\mathcal{A}, E)$ be an Ω-algebra and $u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n)$, briefly $u \approx v$ be an identity in the type of $\mathcal{A}$. We assume, as usual, that variables appearing in terms u and v are from $x_1, \ldots, x_n$.

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Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (i.e., this identity **holds** on $(\mathcal{A}, E)$) if

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E(u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)),$$

for all $a_1, \ldots, a_n \in A$ and the term-operations on $\mathcal{A}$ corresponding to terms u and v respectively.

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An identity $u \approx v$ fulfilled on an algebra $\mathcal{A}$ holds on an Ω-algebra $(\mathcal{A}, E)$ as well.

Theorem

Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all identities in $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p / E_p satisfies the same identities.

Corollary

If a diagonal relation $\Delta_A = \{(a,a) \mid a \in A\}$ *is a cut of* E*, then each identity fulfilled by an* Ω*-algebra* $\overline{\mathcal{A}} = (\mathcal{A}, E)$ *also holds on the underlying algebra* $\mathcal{A}$*.*

Corollary

If a diagonal relation $\Delta_A = \{(a,a) \mid a \in A\}$ is a cut of E, then each identity fulfilled by an Ω-algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ also holds on the underlying algebra $\mathcal{A}$.

By Corollary , we are interested in Ω-algebras which do not contain a copy of the underlying algebra among quotient substructures. An Ω-algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is said to be **proper** if Δ_A is not a cut of E.

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Theorem

$\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a proper Ω-algebra if and only if

there are $a, b \in A$, $a \neq b$, such that $E(a,b) \geqslant \bigwedge\{E(x,x) \mid x \in A\}$.

Proposition

The collection of cuts of E in an Ω-algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a closure system on A^2, a subposet of the weak congruence lattice $\mathrm{Con_w}(\mathcal{A})$ of $\mathcal{A}$.

Proposition

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Theorem (Representation)

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con_w}(\mathcal{A})$ such that

$$\text{if } a \neq b, \text{ then } (a,b) \notin \bigcap \{R \in \mathcal{R} \mid (a,a) \in R\} \quad \text{for all } a, b \in A.$$

Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E.

Sketch of the proof

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We take Ω to be the starting collection $\mathcal{R}$ of weak congruences ordered by the dual of inclusion, $\supseteq$.

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$$E(a,b) := \bigcap (R \in \mathcal{R} \mid (a,b) \in R) \quad \text{for all } a, b \in A.$$

Now we have that $E_R = R$ (the cut determined by R considered as an element of Ω, coincides with R as a weak congruence).

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Now we have that $E_R = R$ (the cut determined by R considered as an element of Ω, coincides with R as a weak congruence).

The structure $(\mathcal{A}, E)$ is then the required Ω-algebra, obtained by the **canonical construction**.

For a symmetric and transitive relation $R \subseteq A^2$, we denote by $\mathrm{dom}R$ the set $\{x \in A \mid (x, x) \in R\}$.

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Corollary

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\mathrm{Con_w}(\mathcal{A})$ fulfilling condition:

$$\text{if } a \neq b, \text{ then } (a,b) \notin \bigcap\{R \in \mathcal{R} \mid (a,a) \in R\} \quad \text{for all } a,b \in A.$$

Let also $\mathcal{F}$ be a set of identities in the language of $\mathcal{A}$ and suppose that for every $R \in \mathcal{R}$, the algebra $\mathrm{dom}R/R$ fulfills these identities. Then there is a complete lattice Ω and an Ω-algebra $(\mathcal{A}, E)$, such that $\mathcal{R}$ consists of cuts of E and $(\mathcal{A}, E)$ satisfies $\mathcal{F}$.

Suppose that we have different complete lattices, Ω_1 and Ω_2 and an algebra $\mathcal{A}$. Let $(\mathcal{A}, E1)$ and $(\mathcal{A}, E2)$ be an Ω_1-valued algebra and an Ω_2-valued algebra respectively. We say that the structures $(\mathcal{A}, E1)$ and $(\mathcal{A}, E2)$ are **cut-equivalent** if their collections of quotient algebras over cuts of $E1$ and $E2$ coincide, i.e., if for every $p \in \Omega_1$ there is $q \in \Omega_2$ such that $\mu 1_p / E1_p = \mu 2_q / E2_q$ and vice versa.

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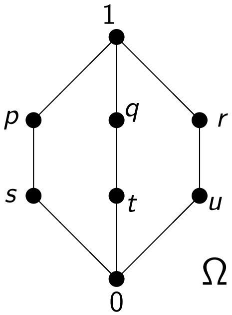
Theorem

Let $\overline{\mathcal{A}} = (\mathcal{A}, E)$ be an Ω-algebra where Ω is an arbitrary complete lattice. Then there is a lattice and a lattice-valued algebra cut-equivalent with $\overline{\mathcal{A}}$, obtained by the canonical construction over $\mathcal{A}$.

Examples

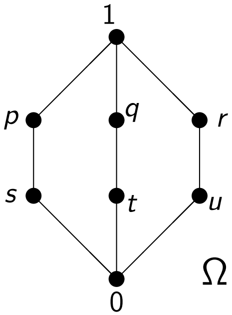
Examples

1. $((\{e, a, b, c, d\}, \cdot, {}'), E)$



Examples

1. $((\{e, a, b, c, d\}, \cdot, ')$, $E)$



$\cdot$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	d	a	a
b	b	c	c	e	a
c	c	c	e	b	b
d	d	d	d	d	e

	e	a	b	c	d
$'$	e	a	c	b	d

E	e	a	b	c	d
e	1	u	t	t	s
a	u	r	0	0	0
b	t	0	q	t	0
c	t	0	t	q	0
d	s	0	0	0	p.

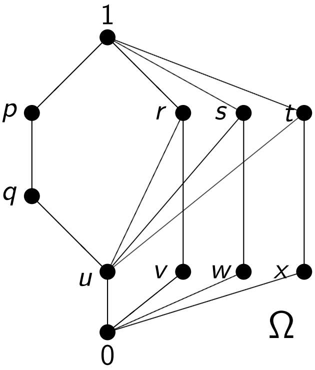
$$\mu = \begin{pmatrix} e & a & b & c & d \\ 1 & r & q & q & p \end{pmatrix}.$$

E	e	a	b	c	d
e	1	u	t	t	s
a	u	r	0	0	0
b	t	0	q	t	0
c	t	0	t	q	0
d	s	0	0	0	p.

$$\mu = \begin{pmatrix} e & a & b & c & d \\ 1 & r & q & q & p \end{pmatrix}.$$

The cuts of E are either diagonal relations on subalgebras (E_q on $\{e, b, c\}$ and E_r on $\{e, a\}$), or they are full relations on one-, two- or three-element subalgebras (e.g., E_t is a full relation on $\{e, b, c\}$). Trivially, E_0 is a full relation on the whole algebra. All the corresponding quotient algebras are groups, hence $(\mathcal{A}, E)$ is an Ω-group. Observe that the basic five-element algebra is not a group.

2.



S_3 – symmetric group, (S_3, E) – the corresponding Ω-group.

$\circ$	e	f	g	h	j	k
e	e	f	g	h	j	k
f	f	e	h	g	k	j
g	g	j	e	k	f	h
h	h	k	f	j	e	g
j	j	g	k	e	h	f
k	k	h	j	f	g	e

E	e	f	g	h	j	k
e	1	x	w	q	q	v
f	x	t	u	0	0	u
g	w	u	s	0	0	u
h	q	0	0	p	q	0
j	q	0	0	q	p	0
k	v	u	u	0	0	r .

$$\mu = \begin{pmatrix} e & f & g & h & j & k \\ 1 & t & s & p & p & r \end{pmatrix}.$$

E	e	f	g	h	j	k
e	1	x	w	q	q	v
f	x	t	u	0	0	u
g	w	u	s	0	0	u
h	q	0	0	p	q	0
j	q	0	0	q	p	0
k	v	u	u	0	0	r .

$$\mu = \begin{pmatrix} e & f & g & h & j & k \\ 1 & t & s & p & p & r \end{pmatrix}.$$

All the structures μ_z/E_z, $z \in \Omega$ are groups of order 3, 2 or 1, hence Abelian.

E	e	f	g	h	j	k
e	1	x	w	q	q	v
f	x	t	u	0	0	u
g	w	u	s	0	0	u
h	q	0	0	p	q	0
j	q	0	0	q	p	0
k	v	u	u	0	0	r .

$$\mu = \begin{pmatrix} e & f & g & h & j & k \\ 1 & t & s & p & p & r \end{pmatrix}.$$

All the structures μ_z/E_z, $z \in \Omega$ are groups of order 3, 2 or 1, hence Abelian.

Therefore, this structure is an Abelian Ω-group, identity

$x \cdot y \approx y \cdot x$

holds as the formula

$\mu(x) \wedge \mu(y) \leqslant E(x \cdot y, y \cdot x)$.

For the cuts, we have e.g., $\mu_p = \{e, h, j\}, \quad \mu_u = \{e, f, g, h, j, k\}.$

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E_p	e	f	g	h	j	k
e	1	0	0	0	0	0
f	0	0	0	0	0	0
g	0	0	0	0	0	0
h	0	0	0	1	0	0
j	0	0	0	0	1	0
k	0	0	0	0	0	0

E_u	e	f	g	h	j	k
e	1	0	0	1	1	0
f	0	1	1	0	0	1
g	0	1	1	0	0	1
h	1	0	0	1	1	0
j	1	0	0	1	1	0
k	0	1	1	0	0	1 .

For the cuts, we have e.g., $\mu_p = \{e, h, j\}$, $\mu_u = \{e, f, g, h, j, k\}$.

E_p	e	f	g	h	j	k
e	1	0	0	0	0	0
f	0	0	0	0	0	0
g	0	0	0	0	0	0
h	0	0	0	1	0	0
j	0	0	0	0	1	0
k	0	0	0	0	0	0

E_u	e	f	g	h	j	k
e	1	0	0	1	1	0
f	0	1	1	0	0	1
g	0	1	1	0	0	1
h	1	0	0	1	1	0
j	1	0	0	1	1	0
k	0	1	1	0	0	1 .

Hence, E_p is a weak congruence on S_3, a diagonal of $\mu_p = \{e, h, j\}$ and μ_p/E_p is a group of order 3.

For the cuts, we have e.g., $\mu_p = \{e, h, j\}$, $\mu_u = \{e, f, g, h, j, k\}$.

E_p	e	f	g	h	j	k
e	1	0	0	0	0	0
f	0	0	0	0	0	0
g	0	0	0	0	0	0
h	0	0	0	1	0	0
j	0	0	0	0	1	0
k	0	0	0	0	0	0

E_u	e	f	g	h	j	k
e	1	0	0	1	1	0
f	0	1	1	0	0	1
g	0	1	1	0	0	1
h	1	0	0	1	1	0
j	1	0	0	1	1	0
k	0	1	1	0	0	1 .

Hence, E_p is a weak congruence on S_3, a diagonal of $\mu_p = \{e, h, j\}$ and μ_p/E_p is a group of order 3. Next, μ_u is the underlying group S_3.

For the cuts, we have e.g., $\mu_p = \{e, h, j\}$, $\mu_u = \{e, f, g, h, j, k\}$.

E_p	e	f	g	h	j	k
e	1	0	0	0	0	0
f	0	0	0	0	0	0
g	0	0	0	0	0	0
h	0	0	0	1	0	0
j	0	0	0	0	1	0
k	0	0	0	0	0	0

E_u	e	f	g	h	j	k
e	1	0	0	1	1	0
f	0	1	1	0	0	1
g	0	1	1	0	0	1
h	1	0	0	1	1	0
j	1	0	0	1	1	0
k	0	1	1	0	0	1 .

Hence, E_p is a weak congruence on S_3, a diagonal of $\mu_p = \{e, h, j\}$ and μ_p/E_p is a group of order 3. Next, μ_u is the underlying group S_3. Therefore, $\mu_u/E_u = \{\{e, h, j\}, \{f, g, h\}\}$ i.e., it is a two-element quotient group, similarly for other cuts.
This Ω-group is obtained by the technique described above.

For the cuts, we have e.g., $\mu_p = \{e, h, j\}$, $\mu_u = \{e, f, g, h, j, k\}$.

E_p	e	f	g	h	j	k
e	1	0	0	0	0	0
f	0	0	0	0	0	0
g	0	0	0	0	0	0
h	0	0	0	1	0	0
j	0	0	0	0	1	0
k	0	0	0	0	0	0

E_u	e	f	g	h	j	k
e	1	0	0	1	1	0
f	0	1	1	0	0	1
g	0	1	1	0	0	1
h	1	0	0	1	1	0
j	1	0	0	1	1	0
k	0	1	1	0	0	1 .

Hence, E_p is a weak congruence on S_3, a diagonal of $\mu_p = \{e, h, j\}$ and μ_p/E_p is a group of order 3. Next, μ_u is the underlying group S_3. Therefore, $\mu_u/E_u = \{\{e, h, j\}, \{f, g, h\}\}$ i.e., it is a two-element quotient group, similarly for other cuts.

This Ω-group is obtained by the technique described above. The closure system i.e., the lattice Ω is $\mathrm{Con_w}(S_3) \setminus \Delta_{S_3}$, consisting of all weak congruences on S_3 except the diagonal Δ_{S_3}. And the order in this lattice is dual to the set inclusion.

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Thanks for listening!