Sums of finitely many distinct reciprocals (work with Donald Silberger)

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Definitions I

Definition

Let $\mathcal{F}$ be the family of finite subsets of $\mathbb{N}$. For any $X \subseteq \mathbb{N}$, let $\mathcal{F}(X)$ be the family of finite subsets of X. Let $\mathcal{I}$ denote the family of finite sets of consecutive natural numbers

$$[m, n] := \{m, m+1, \ldots, n\}$$

Definition

Define the function $\sigma : \mathcal{F} \to \mathbb{Q}^+$ by

$$\sigma S = \sum_{n \in S} \frac{1}{n}$$

Definitions II

Definition

For $r \in \mathbb{Q}^+$, let $\mathcal{F}_r \subseteq \mathcal{F}$ denote the family of finite $S \subseteq \mathbb{N}$ for which $\sigma S = r$.

Definition

We define the functions $\nu : \mathcal{F} \to \mathbb{N}$ and $\delta : \mathcal{F} \to \mathbb{N}$ by $\sigma S = \dfrac{\nu S}{\delta S}$, where νS and δS are coprime.

Example

We let $I = [2, 4] = \{2, 3, 4\}$ and $S = \{3, 4, 5, 7, 12, 20, 42\}$. Then,

$$\sigma I = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{7} + \frac{1}{12} + \frac{1}{20} + \frac{1}{42} = \sigma S$$

We see $\{I, S\} \subseteq \mathcal{F}_{13/12}$ and $\delta I = \delta S = 12$, $\nu I = \nu S = 13$.

Theorems

Theorem 1 (Surjectivity)

For each $r \in \mathbb{Q}^+$, there is an infinite pairwise disjoint family $\mathcal{H}_r \subseteq \mathcal{F}_r$.

Theorem 2 (Injectivity)

Let X be a pairwise coprime subset of $\mathbb{N}$. Then $\sigma \upharpoonright \mathcal{F}(X)$ and $\delta \upharpoonright \mathcal{F}(X)$ are injections. Also, $\sigma C \in \mathbb{N}$ for $C \in \mathcal{F}(X)$ if and only if $C = \{1\}$.

Background I

Theisinger-Kürschák

L. Theisinger [10] proved in 1915 that $\sigma[1,n] \in \mathbb{N}$ *only if* $n = 1$*. In 1918 J. Kürschák [6] proved that* $\sigma[m,n] \in \mathbb{N}$ *only if* $m = n = 1$*.*

Erdös

If $d \geq 1$*, and if either* $m > 1$ *or* $k > 1$*, then* $\sum_{j=0}^{k-1} \frac{1}{m+dj} \notin \mathbb{N}$*. [3]*

Background III

Belbachir and Khelladi

For $\{a_0, a_1, \ldots, a_{k-1}\} \subseteq \mathbb{N}$, *if* $d \geq 1$, *and if either* $m > 1$ *or* $k > 1$, *then* $\sum_{j=0}^{k-1} \frac{1}{(m+dj)^{a_j}} \notin \mathbb{N}$. *[1]*

Obláth

$\sum_{i=m}^{n} \frac{a_i}{i} \notin \mathbb{N}$ *if* i *is coprime to* a_i *for each* $i \in [m, n]$, *where* $[m, n] \neq \{1\}$. *[3]*

Background IV

Erdös -Niven

The function $\sigma|\mathcal{I}$ *is injective.([4], 1946)*

Surjectivity Theorem

Theorem 1

For each $r \in \mathbb{Q}^+$, there is an infinite pairwise disjoint family $\mathcal{H}_r \subseteq \mathcal{F}_r$.

Vital Identity

For all $z \notin \{-1, 0\}$

$$\frac{1}{z} = \frac{1}{z+1} + \frac{1}{z(z+1)}$$

Definition

Define $\diamond : \mathbb{N} \to \mathbb{N}$ and $\star : \mathbb{N} \to \mathbb{N}$ by $\diamond n = n + 1$ and $\star n = n(n+1)$.

Then the Vital Identity says

$$\frac{1}{n} = \frac{1}{\diamond n} + \frac{1}{\star n} \quad \longrightarrow \quad \sigma\{n\} = \sigma\{\diamond n, \star n\}$$

Definition

Let $\mathbf{W}$ denote the set of all finite words $\mathbf{w}$ in the letters $\diamond$ and $\star$. Each word $\mathbf{w} : \mathbb{N} \to \mathbb{N}$ can be thought of an injection under composition of its letters. We let $\mathbf{W}_k \subseteq \mathbf{W}$ be the words of length k in $\mathbf{W}$. For each $n \in \mathbb{N}$ we let $\mathbf{W}\, n$ denote the (perhaps multiset) $\{\mathbf{w}\, n | \mathbf{w} \in \mathbf{W}\}$. We define $\mathbf{W}_k\, n$ analogously.

Example

We can let $\mathbf{w} = \diamond \star \diamond^2 \in \mathbf{W}_4$. Then

$$\mathbf{w}\, 1 = \diamond \star \diamond^2 1 = \diamond \star \diamond 2 = \diamond \star 3 = \diamond 12 = 13 \in \mathbf{W}_4 1 \subseteq \mathbf{W}\, 1$$

Outline of Proof of Surjectivity Theorem (Theorem 1)

Lemma 1

Let $\mathbf{w}\, b = n = \mathbf{w}'\, b$ *with* $\mathbf{w} \neq \mathbf{w}'$. *Then* $|\mathbf{w}| \neq |\mathbf{w}'|$.

Corollary 2

Each $\mathsf{W}_k\, n$ *is a simple set.*

Lemma 3

If k *is a nonnegative integer, then* $\sigma\left(\mathsf{W}_k\, b\right) = \frac{1}{b}$.

Proof of Lemma 3

(basis) $\sigma\left(\mathsf{W}_0\, b\right) = \sigma\{b\} = \frac{1}{b}$.
(ind step) Suppose $\sigma\left(\mathsf{W}_k\, b\right) = \frac{1}{b}$. Note that
$\mathsf{W}_{k+1} = \{\diamond\, \mathbf{w} \mid \mathbf{w} \in \mathsf{W}_k\} \uplus \{\star\, \mathbf{w} \mid \mathbf{w} \in \mathsf{W}_k\}$ and for each $\mathbf{w} \in \mathsf{W}_k$,
$\sigma\{\diamond\, \mathbf{w}\, b, \star\, \mathbf{w}\, b\} = \sigma(\mathbf{w}\, b)$. $\square$

Outline of Proof of Surjectivity, Continued

Corollary 4

Let $b \geq 2$. Then there exists an infinite pairwise disjoint family $\mathcal{G}_b \subseteq \mathcal{F}$ such that $\sigma S = \frac{1}{b}$ for each $S \in \mathcal{G}_b$.

Idea of Proof

We know $\diamond$ and $\star$ are increasing. Choose sequence $k_1, k_2, \ldots$ of integers far enough apart to ensure $\max \mathbf{W}_{k_i} b < \min \mathbf{W}_{k_{i+1}} b$. $\square$

Theorem 1

For each $r \in \mathbb{Q}^+$, there is an infinite pairwise disjoint family $\mathcal{H}_r \subseteq \mathcal{F}_r$.

Proof.

Let $r = \frac{a}{b}$, $b \geq 2$. Let $\mathcal{G}_b = \{S_1, S_2, S_3, \ldots\}$ and for each $n \in \mathbb{N}$ define

$$T_n := \bigcup_{i=0}^{a-1} S_{an+i}$$

□

Injectivity I

Definition

For integers m and n, we write $m|n$ for "m divides n" and $m^v||n$ for "m^v divides n exactly" $\equiv m^v|n$, but $m^{v+1} \nmid n$.

Chebyshev's Theorem (Bertrand's Postulate)

If $n \geq 2$ then there is a prime p such that $n < p < 2n$.

Sylvester's Theorem

If $n < 2m$ then there is a prime $p > n - m$ for which $p|\mathrm{lcm}[m, n]$.

Corollary

If $m < n$ then there is a prime $p > n - m$ such that $p^v||\mathrm{lcm}[m, n]$ for some $v \in \mathbb{N}$, and $p^v||x$ for exactly one $x \in [m, n]$.

Sylvester Powers

Definition

For $X \in \mathcal{F}$, when $v \in \mathbb{N}$ and p is prime, we call p^v a Sylvester power of X if $p^v \| \mathrm{lcm}(X)$ and $p^v \| x$ for exactly one $x \in X$. We will let $S(X)$ denote the set of all Sylvester powers of X.

Example

$$[1000, 1004] = \{1000, 1001, 1002, 1003, 1004\}$$

$$= \{2^3 \cdot 5^3,\ 7 \cdot 11 \cdot 13,\ 2 \cdot 3 \cdot 167,\ 17 \cdot 59,\ 2^2 \cdot 251\}$$

$$S[1000, 1004] = \{2^3,\ 3,\ 5^3,\ 7,\ 11,\ 13,\ 17,\ 59,\ 167,\ 251\}$$

Lemma 1

For a prime p, if $p^v \| \mathrm{lcm}(X)$ *while* $p^v > \max X - \min X$ *then* $p^v \in S(X)$ *and if* $2^v \| \mathrm{lcm}[m, n]$ *then* $2^v \in S[m, n]$.

Lemma 2

For $X \in \mathcal{F}$ *and p prime and* $v \in \mathbb{N}$, *let* $p^v \in S(X)$. *Then* $p^v \| \delta X$.

Corollary: Theisinger-Kürschák

$\sigma[m, n] \in \mathbb{N}$ *only if* $m = n = 1$.

Injectivity III

Theorem

For $\{X, Y\} \subseteq \mathcal{F}$ and $v \in \mathbb{N}$, let $p^v \in S(X) \setminus S(Y)$ with $p^v > \max Y - \min Y$. Then $\delta X \neq \delta Y$ and so $\sigma X \neq \sigma Y$.

Proof of Theorem

By Lemma 3, $p^v \| \delta X$.
There exists $u \geq 0$ such that $p^u \| \operatorname{lcm}(Y)$. Since $p^v \notin S(Y)$ and $p^v > \max Y - \min Y$, $u \neq v$.
If $u > v$, then $p^u \in S(Y)$ and so $p^u \| \delta Y$. So $\delta Y \neq \delta X$.
If $u < v$, then the biggest power of p that divides δY is less than p^v and so $\delta Y \neq \delta X$. □

Proof of Theorem 2

Theorem 2

Let X be a pairwise coprime subset of $\mathbb{N}$. Then $\sigma\restriction\mathcal{F}(X)$ and $\delta\restriction\mathcal{F}(X)$ are injections. Also, $\sigma C \in \mathbb{N}$ for $C \in \mathcal{F}(X)$ if and only if $C = \{1\}$.

Proof of Theorem 2

Let $X \subseteq \mathbb{N}$ be pairwise coprime and $\{A, B\} \in \mathcal{F}(X)$ with $A \neq B$. Then WLOG there is $a \in A \setminus B$ and a prime p that divides a but does not divide any element of $(A \setminus \{a\}) \cup B$. Then some power of p is in $S(A)$ and hence divides δA, but p does not divide δB. Thus, $\delta A \neq \delta B$ and so $\sigma A \neq \sigma B$. □

Questions

Question 1

Does there exist an $X \subseteq \mathbb{N}$ such that $\sigma \restriction \mathcal{F}(X)$ is bijective onto $\mathbb{Q}^+$?

Question 2

If $1 < m < n < m' < n'$ and if $n - m \leq n' - m'$, can we conclude $S[m,n] \neq S[m',n']$?

Further Reading I

- [1] H. Belbachir and A. Khelladi, *On a sum involving powers of reciprocals of an arithmetic progression*, Ann. Mathematicae et Informaticae **34** (2007), 29-31.
- [2] P. Erdös, *Egy Kürschák-Féle Elemi Számelméleti Tétel Általánosítása*, Matematikai és Fizikai Lapok BD. XXXIX, Budapest (1932), 1-8.
- [3] P. Erdös, *A theorem of Sylvester and Schur*, J. London Math. Soc. **9** (1934), 191-258.
- [4] P. Erdös and I. Niven, *Some properties of partial sums of the harmonic series*, Bull. Amer. Math. Soc. **52** (1946), 248-251.
- [5] P. Hoffman, "The Man Who Loved Only Numbers: The Story of Paul Erdös and the Search for Mathematical Truth", N. Y. Hyperion, 1998.

Further Reading II

[6] J. Kürschák, Matematikai és Fizikai Lapok, **27** (1918), 299.

[7] T. N. Shorey, *Theorems of Sylvester and Schur*, Math. Student (2007), Special Centenary Volume (2008), 135-145. Online article (http://www.math.tifr.res.in/shorey/newton.pdf).

[8] I. Schur, *Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen. II.* Sitzungsber. Preuss. Akad. Wiss. Berlin Phys. Math. K1. **14** (1929), 370-391.

[9] J. J. Sylvester, *On Arithmetical Series*, Messenger of Math. **21** (1892), 1-19, 87-120 (Collected Mathematical Papers, Bd. **14**, 687-731).

[10] L. Theisinger, *Bemerkung über die harmonische Reihe*, Monatshefte für Mathematik und Physik **26** (1915), 132-134.