Some topological lattices.

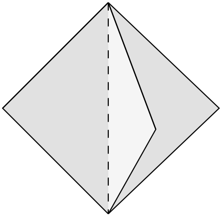
Walter Taylor

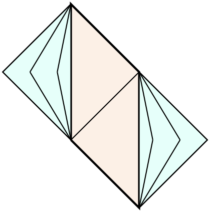
Algebras Lattices in Hawaii 2018
Honoring R. Freese, W. Lampe, J. B. Nation

May 24, 2018

Our subject matter.

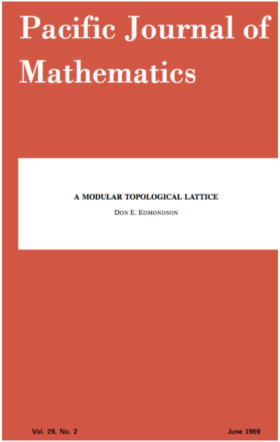
We will be concerned with some finite two-dimensional simplicial complexes, such as





(Three triangles, then eight.) We will exhibit lattice equations that are continuously satisfiable on some but not all of such spaces. Results phrased in terms of lattice homomorphisms.

The topological lattice $\Delta(M_3)$ and its antecedents



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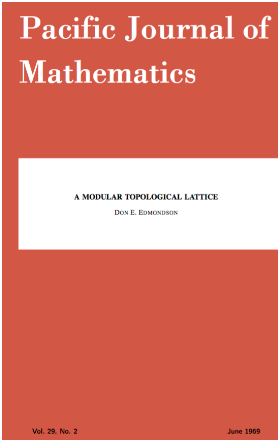
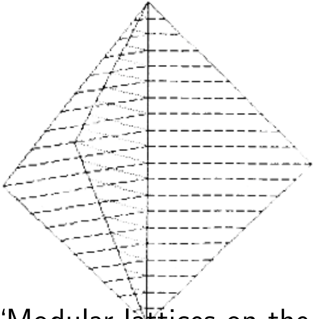
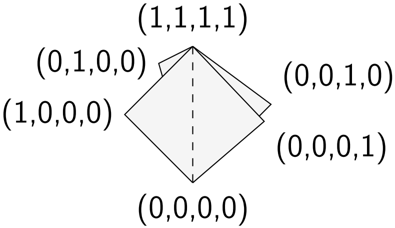


G. Gierz and A. Stralka, AU 1989



"Modular lattices on the 3-cell are distributive"

The spaces $\Delta(M_n)$



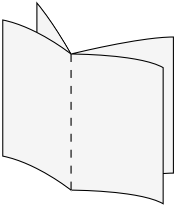
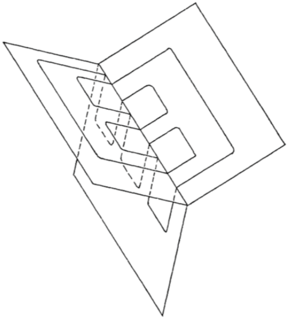


Figure: The topological space $\Delta(M_4)$ — the 4-book

Warning: no particular page order.

BTW people have studied book spaces. E.g. knots:

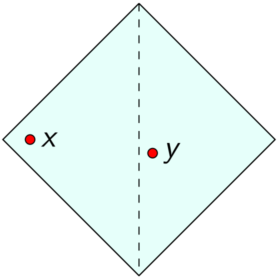
A trefoil knot in $\Delta(M_3)$



Persinger, PJM, 1966

How to make $\Delta(M_n)$ into a lattice.

Each pair of flaps forms a sublattice K isomorphic to $[0,1]^2$. (With $(1,1)$ at the top and $(0,0)$ at the bottom.) If x and y belong to the two flaps, their join is as in $[0,1]^2$.



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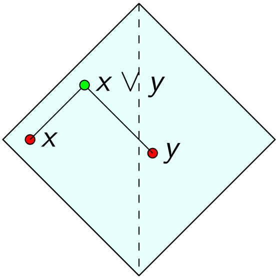

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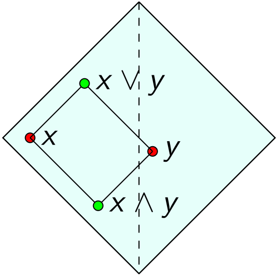

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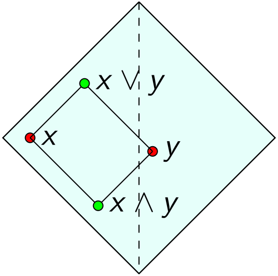
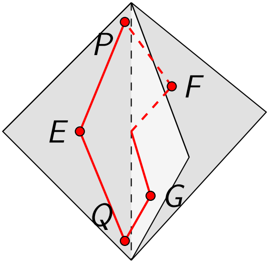


When do we have $x \wedge y = 0$?

Why this lattice on $\Delta(M_n)$ is modular.

Given the red pentagon $QEPFG$ in $\Delta(M_3)$:



So take two new green points, S and T, with $G \leq T \leq S \leq F$, along the segment rising from G to F. Now $\{E, S, T\}$ generates a pentagon inside a two-flap sublattice.

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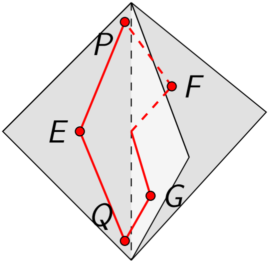

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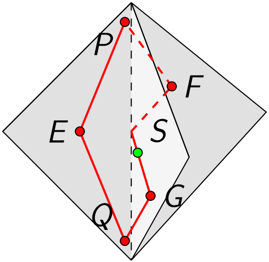

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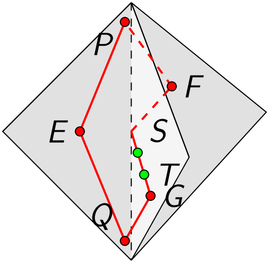

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Internet proof that $\Delta(M_3)$ cannot be distributive

A search on "distributive topological lattice compact" quickly took me to this conclusion of K. Baker and A. Stralka, 1970:

We are now able to prove and generalize a conjecture first formulated by Dyer and Shields [7, p. 447] and sharpened by Anderson [2, p. 62].

COROLLARY 3.3. *Let L be a metrizable, distributive topological lattice of finite breadth n which is either* (a) *compact or* (b) *locally compact, separable, and connected. Then L can be embedded, topologically and algebraically, in the n-cell I^n.*

This led me to look up "breadth lattice topological," and here is what came up first:

Internet proof, continued

T. H. Choe, 1969:

THE BREADTH AND DIMENSION OF A TOPOLOGICAL LATTICE

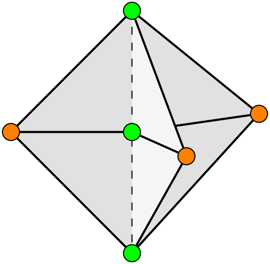
TAE HO CHOE

E. Dyer and A. Shields [**7**] conjectured that if L is a compact connected metrizable distributive topological lattice, then $\dim(L)$ is equal to the breadth of L. L. W. Anderson [**1**] has proved that if L is a locally compact, (chain-wise) connected distributive topological lattice then the breadth of L is less then or equal to the codimension of L.

In this note we shall show that if L is a locally compact connected distributive topological lattice of inductive dimension n and if the set of points at which L has dimension n has nonvoid interior, then the breadth of L is also n.

A Kuratowski (*et al.*) forbidden graph, in $\Delta(M_3)$

$K_{3,3}$ (3 houses, 3 utilities) — sketch courtesy of G. Bergman:



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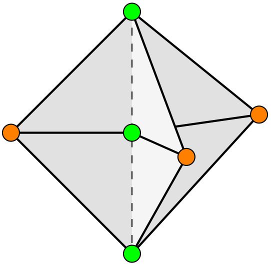

Thus the topological space $\Delta(M_3)$ is not embeddable in $\mathbb{R}^2$.

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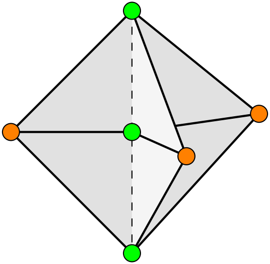

Thus the topological space $\Delta(M_3)$ is not embeddable in $\mathbb{R}^2$.
Contradiction.

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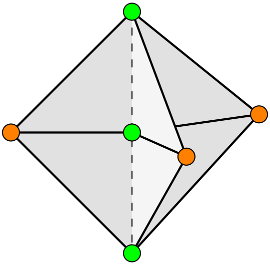
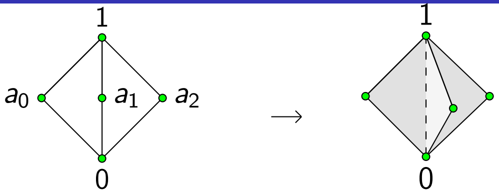


Thus the topological space $\Delta(M_3)$ is not embeddable in $\mathbb{R}^2$. Contradiction. So there are no continuous lattice operations making $\Delta(M_3)$ into a distributive lattice. [Or use trefoil knot.]

A (0,1)-homomorphism from M_3 to $\Delta(M_3)$.



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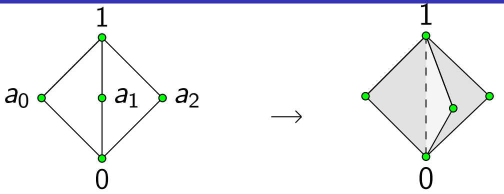

In general $M_n \longrightarrow \Delta(M_n)$, but $M_{n+1} \not\longrightarrow \Delta(M_n)$. Here we mean, **there are no continuous lattice operations on the space $\Delta(M_n)$ admitting a (0,1)-homomorphism from $\mathbf{M}_{n+1}$.**

W Taylor, AU 78 (2017), 601–612.

Proof that $M_{n+1} \not\longrightarrow \Delta(M_n)$.

Easy fact: in our topological lattice $\Delta(M_n)$, if $a \wedge b = 0$ with $a \neq 0$, $b \neq 0$, or dually, then a and b both lie on the periphery of $\Delta(M_n)$. (Remember that the meet takes place in a sublattice isomorphic to $[0, 1]^2$.)

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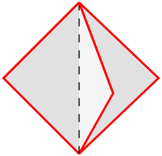
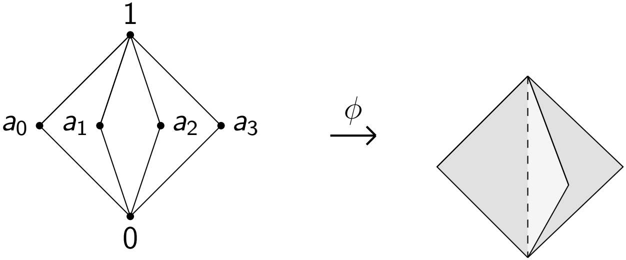
Theorem (more difficult). *This holds for any compatible lattice structure on the space $\Delta(M_n)$.* — J W Lea, Jr. (1973) — Proof: algebraic topology.



(This is true for a large class of topological spaces.)

Proof that $M_{n+1} \not\longrightarrow \Delta(M_n)$, continued

If we had a $(0,1)$-homomorphism ϕ into a topological lattice:



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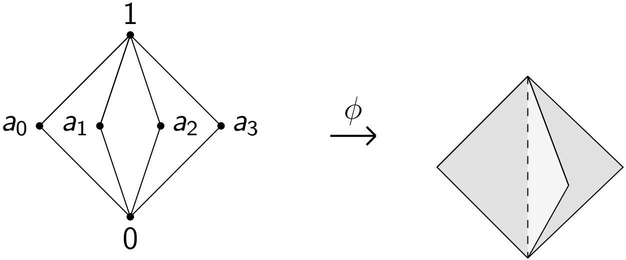

then the four sets $[0, \phi(a_i)] \subseteq \Delta(M_3)$ would be non-trivial connected subsets of the periphery, disjoint except for one point in common. It is almost obvious that four such sets do not exist in the periphery of $\Delta(M_3)$. ∎

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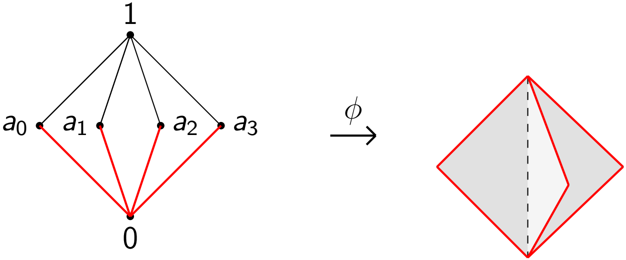

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Alternate viewpoint; one motivation for this work.

Let A be a topological space. (1) There exist continuous lattice operations $\wedge$ and $\vee$ on A and a $0, 1$-homomorphism $\phi\colon \mathbf{M}_n \longrightarrow \langle A, \wedge, \vee \rangle$, iff (2) the space A is compatible with these identities:

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- *Axioms, in* $\wedge$, $\vee$, 0 *and* 1, *for lattice theory with 0 and 1;*
- $a_i \vee a_j \;\approx\; 1$ *(for* $0 \leq i < j < n$*);*
- $a_i \wedge a_j \;\approx\; 0$ *(for* $0 \leq i < j < n$*).*

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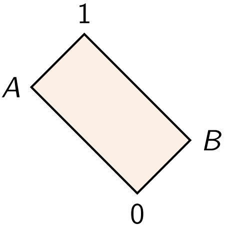
Of these two, we shall routinely use the viewpoint (1).

Assembly of a space K_n^m from $[0, 2] \times [0, 1]$



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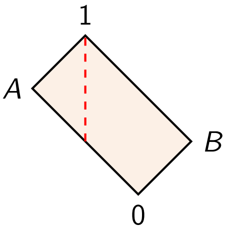

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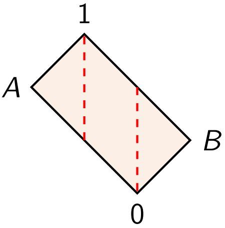

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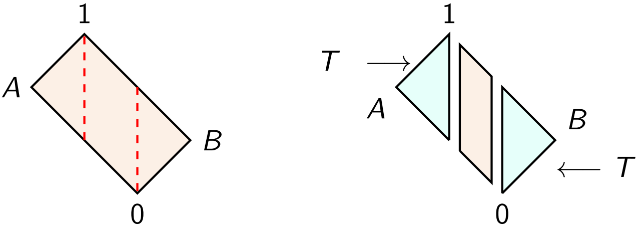

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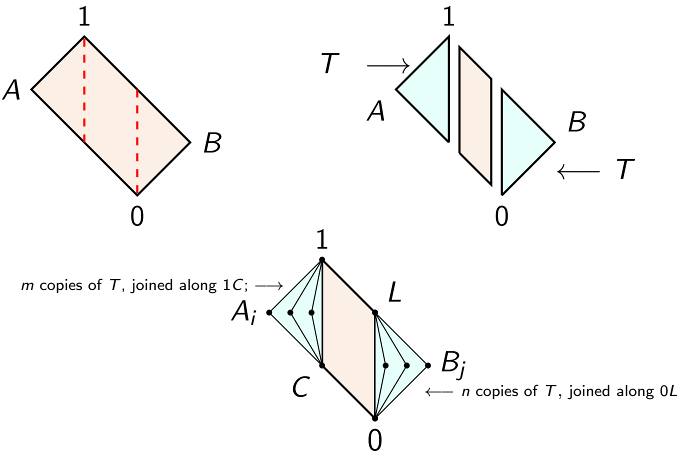
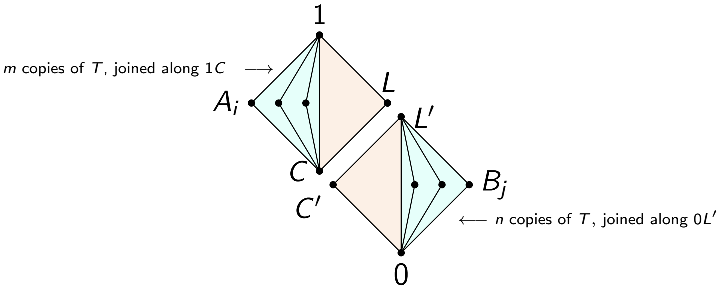


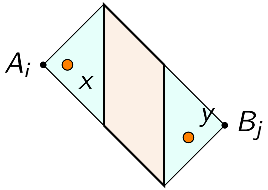
Making the space K_n^m into a topological lattice $\mathbf{K}_n^m$

A Hall-Dilworth gluing of $\Delta(M_{m+1})$ and $\Delta(M_{n+1})$:



Another view of the lattice operations on K_n^m

We illustrate $x \vee y$ for $x \in A_i$ and $y \in B_j$. We take our rectangular picture of K_n^m, making sure that A_i and B_j are at the top of their respective stacks. Then we join x and y by intersecting upward lines parallel to the sides of the rectangle.



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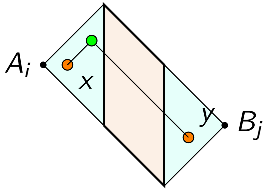

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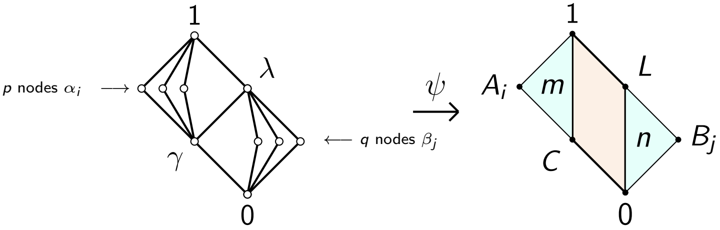
The role of the finite lattice $\mathbf{M}_q^p$.

We build a corresponding finite lattice by gluing M_p and M_q:



Theorem. *There exist continuous lattice operations on the space K_n^m, and a (0,1)-homomorphism ψ as indicated, if and only if $p \vee q \leq m \vee n$ and $p \wedge q \leq m \wedge n$.*

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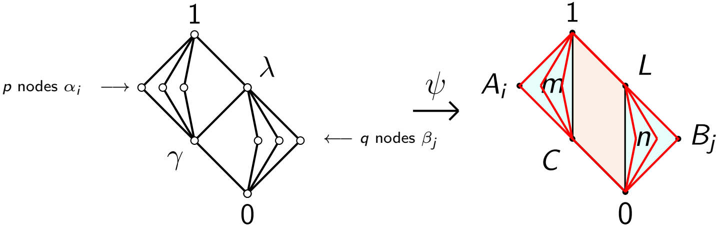

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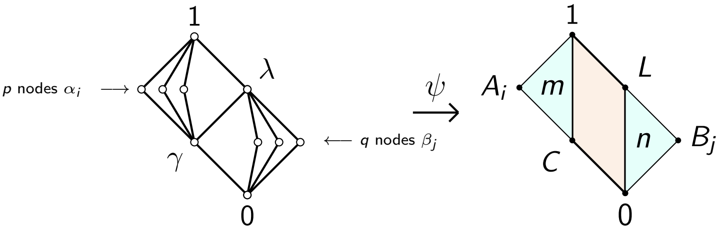
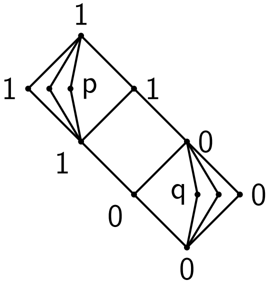
Proof of "only if": the periphery argument from before. (In this direction, there is no assumption that the points α_i, β_j, γ, . . . , map to A_i, B_j, C, ...)

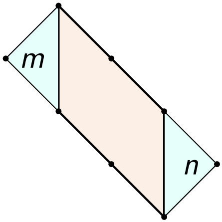
The corresponding compatibility/incompatibility result.



Equivalently, *The space K_n^m topologically models Σ_q^p ((0,1)-lattice theory plus equations defining the finite lattice $\mathbf{M}_{p,q}$) if and only if $p \vee q \leq m \vee n$ and $p \wedge q \leq m \wedge n$.*

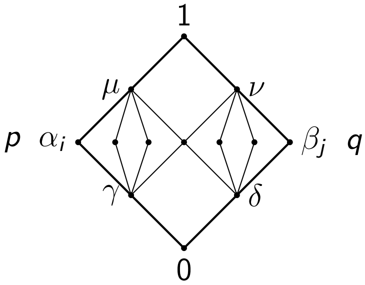
Stretching K_n^m and $\mathbf{M}_{p,q}$.

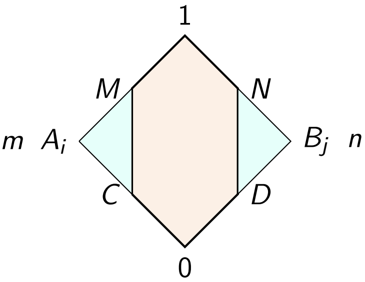




The finite lattice here maps onto a two-element lattice (as indicated by the labels); hence it maps into any topological lattice; hence it does not contribute to our discussion. Clearly the previous example was more interesting since it is simple.

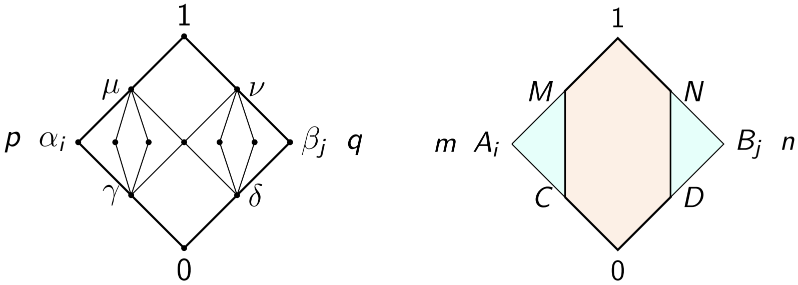
A symmetric example.





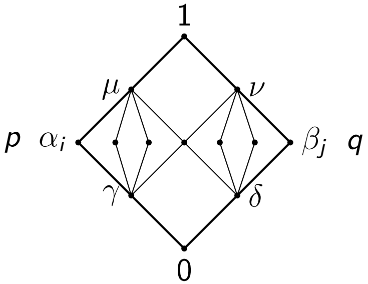
Comments.

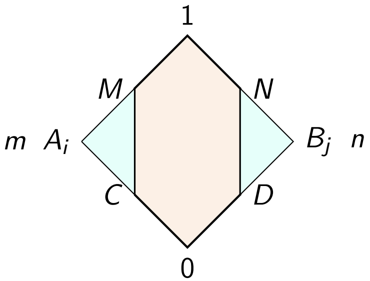
A symmetric example.



Comments.(1) The space here is homeomorphic to the space K_n^m that we already examined, but the two finite lattices are non-isomorphic.

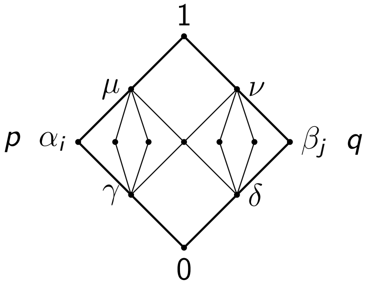
A symmetric example.

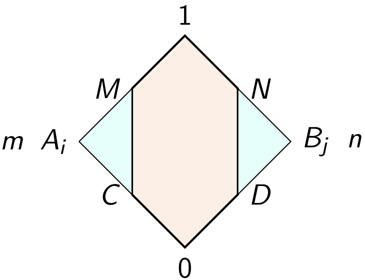




Comments.(1) The space here is homeomorphic to the space K_n^m that we already examined, but the two finite lattices are non-isomorphic.(2) Our lattice here has exactly two non-trivial homomorphic images: $\mathbf{M}_p$ and $\mathbf{M}_q$.

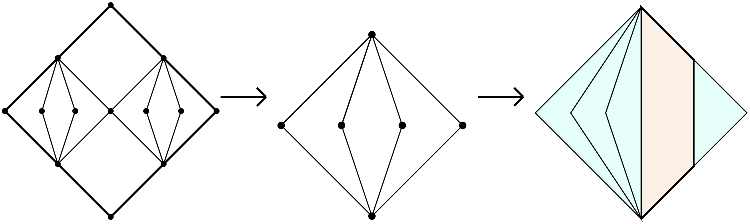
A symmetric example.



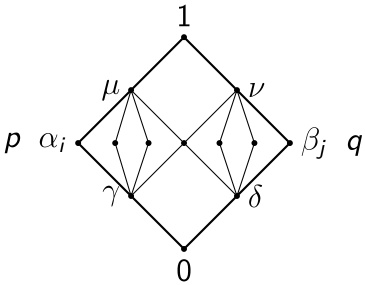


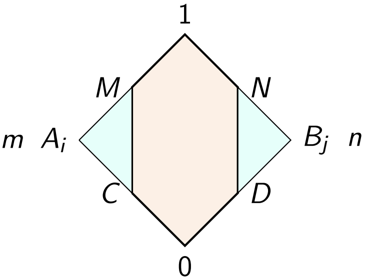
Comments.(1) The space here is homeomorphic to the space K_n^m that we already examined, but the two finite lattices are non-isomorphic.(2) Our lattice here has exactly two non-trivial homomorphic images: $\mathbf{M}_p$ and $\mathbf{M}_q$. (3) This arises from a H-D gluing.

So we must also consider this possibility:



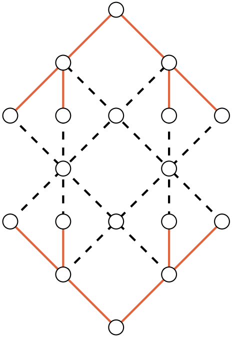
Analysis of the symmetric example.

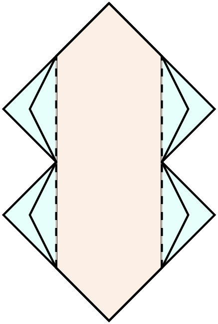




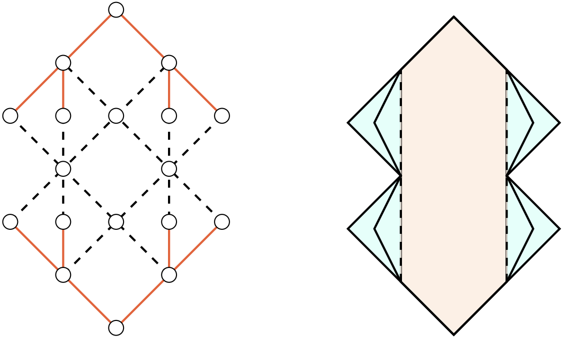
Theorem *This space can be made into a topological lattice so that there is a homomorphism from the finite lattice to the topological lattice, iff* $p \wedge q \leq m \vee n$.

Previous example doubled (drawn $p = q \cdots = 2$)



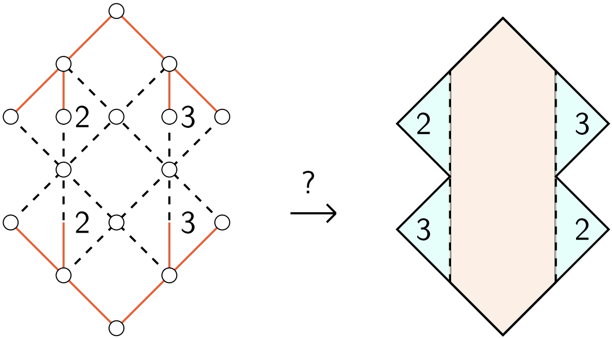


Previous example doubled (drawn $p = q \cdots = 2$)



Lea's Theorem does not force the four central points here to map to the periphery, so only the colored lines indicate intervals that must lie in the periphery.

Thus this possiblity remains unsettled:



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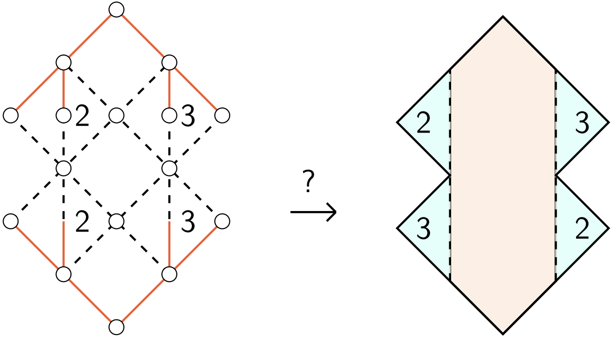
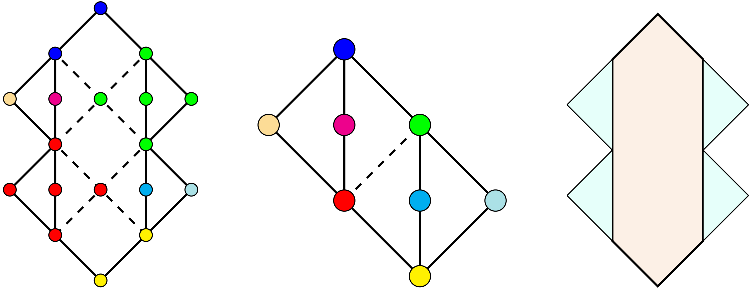


Our usual method of construction fails here. Meanwhile our necessary condition (having the right configuration exist in the periphery) is still satisfied.

Another way of possibly defining a hom here:



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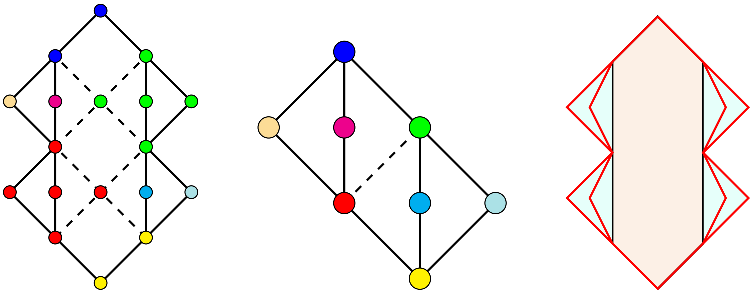

Open questions

0. Does there exist a space that can be a topological lattice, but not a modular topological lattice? Can it be a finite simplicial complex?

1. How far can this go? Full analysis of the topology?

2. Are there results for spaces with a 2-D periphery? (Lea's Theorem is still valid.)

3. Strengthening of Lea's Theorem?

4. General theory of homomorphism Finite Lattice $\longrightarrow$ Top Lattice? Or more generally, Finite Algebra $\longrightarrow$ Topological Algebra? Or of finite sublattices of topological lattices?