UCALC and testing for Maltsev conditions

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Dedication

This talk is dedicated to Bill, J.B., and Ralph for the example that they set by embracing the computer, way back in the mid 1980's, as a vital tool for conducting research in our field.

History and acknowledgments

The first version of the program has been written by students of Matt Valeriote in 1988-91, for XWindows. This version had many of the features of the present program, and some additional ones, too, which we plan to implement. These include working with terms and identities, drawing subalgebra lattices, etc. However that version has many bugs. It is available from

ftp://icarus.math.mcmaster.ca:/pub/UA/Algebra.tar.gz

Many of the ideas for writing the present program came from that version. This old version had some partial MS-DOS ports, but these are now made obsolete by the present program.

The other source of inspiration has been Ralph Freese's lattice drawing program, which is integrated into the present program. It can display any lattice, not just congruence lattices. You can try out the original version online, or download the source. The address is

http://www.math.hawaii.edu/~ralph/LatDraw

The program makes use of Ralph Freese's algorithms concerning partitions, to speed up calculations. You can download the corresponding reprints from

http://www.math.hawaii.edu/~ralph/papers.html

** From `http://www.math.hawaii.edu/~ralph/software/uaprog/oldversion.html`

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- UACALC can do this for us.

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- The previous algebra was constructed sometime in 2004 while working on special cases of the Constraint Satisfaction Problem.
- Usually, the algebras were designed to satisfy a given Maltsev condition, but it wasn't so clear if they also, incidentally, satisfied others.
- Back then, the only algorithms that were implemented by UACALC involved building free algebras, and so could not be used for most algebras, especially those that had rich local structure.

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- This decision problem was shown to be hard by Bergman, Juedes, and Slutzki in the paper "Computational complexity of term-equivalence". Harvey Friedman is also credited with proving this result.

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- *Does* $\mathbf{V}(\mathbb{A})$ *omit the unary and semilattice types?*

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Theorem

Let $\mathbb{A}$ be a finite idempotent algebra. Then $\mathbf{V}(\mathbb{A})$ omits the unary type if and only if some 2-generated subalgebra of $\mathbb{A}$ has a 2-element quotient that is term equivalent to a set.

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- $\mathbf{V}(\mathbb{A})$ *is congruence n-permutable for some n.*

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- An alternate characterization that is used by UACALC to perform this test is: for all x and y in $\mathbb{A}$, if $a = (x, x)$, $b = (x, y)$ and $c = (y, x)$, then in the subalgebra $\mathbb{B}$ of $\mathbb{A}^2$ generated a, b and c

$$(a, c) \in \beta \vee (\alpha \wedge \gamma),$$

where $\alpha = \mathrm{Cg}^{\mathbb{B}}(a, c)$, $\beta = \mathrm{Cg}^{\mathbb{B}}(a, b)$, and $\gamma = \mathrm{Cg}^{\mathbb{B}}(b, c)$.

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  where $\alpha = \mathrm{Cg}^{\mathbb{B}}(a, c)$, $\beta = \mathrm{Cg}^{\mathbb{B}}(a, b)$, and $\gamma = \mathrm{Cg}^{\mathbb{B}}(b, c)$.
- The runtime of the associated algorithm can be bounded be a polynomial of degree 4 in the size of $\mathbb{A}$.

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Remark

We showed that in this situation, if some minimal set of some algebra in $\mathbf{V}(\mathbb{A})$ has a non-empty tail, then we could find such an algebra as a 3-generated subalgebra of $\mathbb{A}^2$.

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Theorem

Let $\mathbb{A}$ be a finite idempotent algebra. Then $\mathbb{A}$ has a Maltsev term if and only if it has enough local Maltsev terms:
for each a, b, c, $d \in A$, there is a term operation $t_{(a,b,c,d)}(x, y, z)$ such that

$$t_{(a,b,c,d)}(a, b, b) = a \quad \text{and} \quad t_{(a,b,c,d)}(c, c, d) = d.$$

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- Since subalgebra generation is linear (sort of) in the size of the algebra, then the above leads to an algorithm for testing for a Maltsev term whose runtime can be bounded by a polynomial of degree 6 in the size of the algebra.

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- *[Kazda] having a k-ary weak near unanimity term.*

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- *[Barto, Kozik] having a k-ary cyclic term.*
- *[Kazda] having a k-ary weak near unanimity term.*
- *[Val.] having a sequence of Jónsson terms of length 4.*

More local term results

- A number of other well known strong idempotent Maltsev conditions can also be characterized via suitable local term conditions, for finite idempotent algebras.

Theorem

For a finite idempotent algebra $\mathbb{A}$, the following properties can be characterized using local terms, and hence can be tested in polynomial-time: for a fixed value of k,

- *[Horowitz] having a k-ary near unanimity term.*
- *[Horowitz] having a k-edge term.*
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- *[DeMeo, Freese, Val.] having a difference term.*

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- When $\mathbb{A}$ is idempotent, then we only need to perform the following test: for all x and y in $\mathbb{A}$, the tuple (x, x, x, x) is in the subalgebra of $\mathbb{A}^4$ generated by

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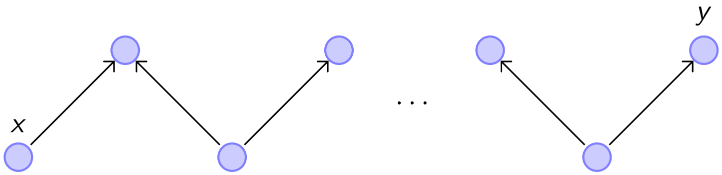
- Let's try this on our demo algebra.

A general result

- Alex Kazda and I have been able to unify several of the results and arguments from the previous theorem to show that a fairly large class of idempotent Maltsev conditions can be decided in polynomial-time for finite idempotent algebras.



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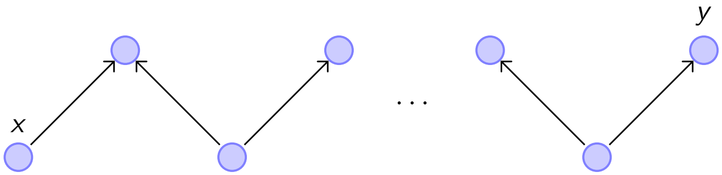

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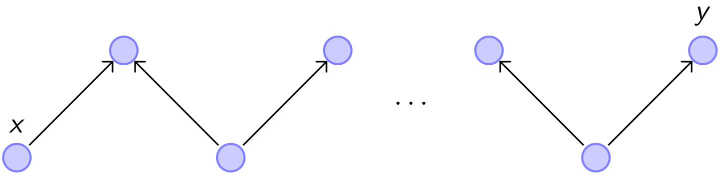
- For example, the following path captures the property of having a sequence of Jónsson terms of length equal to the length of the path.



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- *The polynomial-time algorithms that we produce are based on the "local terms" method.*
- *We show that a path definable Maltsev condition will hold for a finite idempotent algebra if and only if the algebra has enough local term versions of the Maltsev condition.*

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Conjecture (Kazda, Val.)

If Σ is a strong, idempotent, linear Maltsev condition, then there is a polynomial-time algorithm to decide if a given finite idempotent algebra satisfies it.

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- *A* *minority term* *is a ternary term* $m(x, y, z)$ *that satisfies:*

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Theorem (Freese, Nation, Val.)

Testing for a semilattice term, even for finite idempotent algebras, is an EXP-TIME complete problem.

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